Schubert Calculus and the Cohomology of the Grassmannian

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- Motivating Examples
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- Projective Geometry

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- Definitions
- Properties of Tableau
- Symmetric Polynomials

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- Schubert Cells
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- **???**: How many k-dimensional subspaces of $\mathbb{C}^n$ intersect each of the $k(n-k)$ fixed subspaces of dimension $(n-k)$ nontrivially?

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- With incidence combinatorics, enumerative geometry, and the advent of Schubert Calculus, we can consider this problem solved.
- Let us see a few theorems and problems in classical incidence combinatorics.

Theorems and Examples

- **Bezout's Theorem** If a degree d curve γ and a degree d' curve γ' have no common component, then they intersect in at most dd' points (and if the underlying field k is algebraically closed, one works projectively, and one counts intersections with multiplicity, they intersect in exactly dd' points).

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  The last one has a very elegant proof!

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- $Gr(1, n)$ corresponds to the projective spaces $\mathbb{P}^n$, so it is a generalization.
- It has the structure of a differentiable complex manifold.
- Our intersection questions can be nicely 'projectivized' (e.g. we want all lines to intersect - gives us precise values from Bezout's Theorem).

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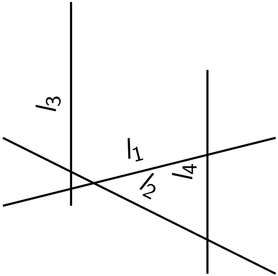

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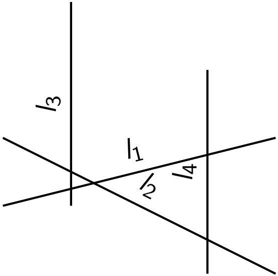

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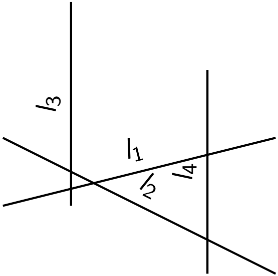

- We want to construct the following sets:
  $\Omega_i = \{$ All lines that pass through $l_i\}$.
  And then compute the intersection $|\Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4| \stackrel{?}{=} 2$.

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- We will focus on the last three topics listed above.

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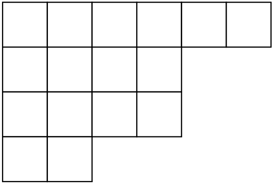

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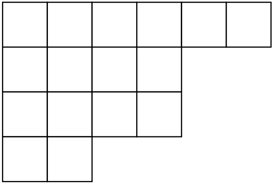

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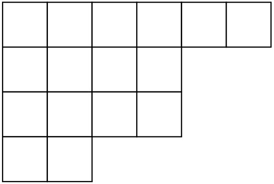

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- The diagram above has the shape $\lambda = (6, 4, 4, 2)$.

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Standard Tableau

- If the entries we use are numbers from 1 to n, where $n = |\lambda| = 6 + 4 + 4 + 2 = 16$ in this case, then our tableau is a standard tableau.

Semistandard Young Tableau (SSYT)

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A **skew shape** *is the difference ν/λ formed by cutting out the Young diagram of a partition λ from the strictly larger partition ν. A skew shape is a* **horizontal strip** *if no column contains more than one box.*

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A **semistandard Young tableau (SSYT)** *of shape ν/λ is a Standard filling of the skew shape ν/λ. An SSYT has* **content** *μ if there are μ_i boxes labeled i for each i. The* **reading word** *of the tableau is the word formed by concatenating the rows from bottom to top.*

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- The following is a semistandard Young tableau of shape ν/λ and content μ where $\nu = (6, 5, 3)$, $\lambda = (3, 2)$, and $\mu = (4, 2, 2, 1)$. Its reading word is 134223111.

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- I want to give a small introduction to their combinatorics.

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Theorem (Hook length formula)

The number of standard Young tableaux of shape λ is

$$\frac{|\lambda|!}{\prod_{s\in\lambda} hook(s)}$$

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- The empty Tableau, denoted by ϕ, is a unit in this monoid.

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- Take this entry that was bumped from the first row, and repeat the process on the second row.
- Keep going until the bumped entry can be put at the end of the row it is bumped into, or until it is bumped out the bottom, in which case it forms a new row with one entry.

Example of the bumping algorithm

For example, if we want to insert 2 into the following Tableau T:

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4	4	6	
4	6		

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$\leftarrow 2$

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$\leftarrow 3$

Example of the bumping algorithm (contd.)

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Example of the bumping algorithm (contd.)

1	2	2	2
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$\leftarrow 5$

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$T \leftarrow 2$

- The algorithm can be run backwards to obtain the original tableau T, if we have the position of the additional box.
- We can now generalize to find the product of two tableaux, $T \cdot U$.

Example of the bumping algorithm (contd.)

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 2 \\ \hline 2 & 3 & 3 & 5 \\ \hline 4 & 4 & 6 \\ \cline{1-3} 5 & 6 \\ \cline{1-2} \end{array}$$

$\leftarrow 5$

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 2 \\ \hline 2 & 3 & 3 & 5 \\ \hline 4 & 4 & 5 \\ \cline{1-3} 5 & 6 & 6 \\ \cline{1-3} \end{array}$$

$T \leftarrow 2$

- The algorithm can be run backwards to obtain the original tableau T, if we have the position of the additional box.
- We can now generalize to find the product of two tableaux, $T \cdot U$.
- List the entries of U in order from left to right, and from bottom to top, and then row-bump into T, in order.

Words: the Plactic Monoid

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- It can then be shown that every word is Knuth equivalent to the word of a unique tableau.
- The monoid of tableaux is isomorphic to the monoid of Knuth equivalent words (called the plactic monoid), with word juxtaposition as the associative product, i.e. $w \cdot v = wv$.
- Any monoid has an associated ring with it. We call this the **tableau ring**, $R_{[m]}$ - this is a free $\mathbb{Z}$-module with basis the tableau with entries in the alphabet $[m]$.

Tableau Definition of Schur Functions

Theorem

There is a canonical homomorphism from the tableau ring, $R_{[m]}$ onto the ring $\mathbb{Z}[x_1, x_2, \ldots, x_m]$ of polynomials.

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- We want to construct rules to multiply two Schur polynomials using this machinery.
- Two important special cases are of interest to us in Schubert Calculus - which follow directly from properties of row-bumping.
- They are the same as formulas found by Pieri for multiplying Schubert varieties in the intersection (cohomology) ring of a Grassmannian.

Symmetric Polynomials

Definition

The (graded) **ring of symmetric functions** $\Lambda_{\mathbb{C}}(x_1, x_2, \dots)$ *is the ring of bounded-degree formal power series* $f \in \mathbb{C}[[x_1, x_2, \dots]]$ *which are symmetric under permuting the variables, that is,* $f(x_1, x_2, \dots) = f(x_{\pi(1)}, x_{\pi(1)}, \dots)$ *for any permutation* $\pi : \mathbb{Z}^+ \to \mathbb{Z}^+$ *and* $deg(f) < \infty$.

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- For instance, $x_1^2 + x_2^2 + x_3^2 + \dots$ is a symmetric function of degree 2.
- Pieri Formulas With (p) and (1^p) the Young diagrams with one row and one column of length p, where the sum is over all μ that come from adding p boxes to λ, no two in the same column/row:

$$S_\lambda \cdot S_{(p)} = \sum_\mu S_\mu$$

$$S_\lambda \cdot S_{(1^p)} = \sum_\mu S_\mu$$

Constructing Schur Polynomials from Tableaux

Definition

Given a semistandard Young tableau T of shape λ, define $x^T = x_1^{m_1} x_2^{m_2} \dots$ where m_i is the number of i's in T. The Schur polynomial for a partition λ is defined by

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where the sum ranges over all SSYT's T of shape λ.

- In fact, the correspondence described above between the Tableau ring and the polynomial ring is similar to this description.
- **Fact** Schur polynomials are symmetric.
- **Fact** They form a vector space basis of $\Lambda(x_1, x_2, \dots)$ as λ ranges over all partitions.

Example of a Schur Polynomial

For the partition $\lambda = (2, 1)$, the tableaux

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \ \cdots$$

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The corresponding Schur polynomial is then

$$s_\lambda = x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + \ldots$$

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4 Intersections of the Schubert Varieties
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Motivation

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- This will help us develop the connection between the Schubert Varieties and the Cohomology ring.

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An n-cell is a topological space homeomorphic to the open ball $|v| < 1$ in $\mathbb{R}^n$, and its associated n-disk is its closure $|v| \leq 1$ in $\mathbb{R}^n$.

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- Continue this process, inductively building the n-skeleton X^n, and set $X = X^n$.
- There are some topological caveats you need to be careful about (e.g. infinite skeletons, weak/coherent topology) but these will not be bothersome to us.

CW Complex Construction (contd.)

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- We then set $X^n = X^{n-1} \coprod_\alpha e^n_\alpha$, where each e^n_α is an open n-disk.
- If we stop at a finite stage $n < \infty$, then we set $X = X^n$ as our CW complex. If we continue indefinitely, then we set $X = \coprod_n X^n$, and X is given the weak topology: A set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n, and the topology on X^n is the usual Euclidean topology.

Examples of CW Complexes

- The sphere S^n has the structure of a cell complex with just two cells, e^0 and e^n, the n-cell being attached by the constant map $S^{n-1} \to e^0$. This is equivalent to regarding S^n as the quotient space $D^n/\partial D^n$.

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- $\mathbb{R}P^n$ is the quotient space of a hemisphere D^n with antipodal points of ∂D^n identified. Since ∂D^n with antipodal points identified is just $\mathbb{R}P^{n-1}$, we see that $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n-cell, with the quotient projection $S^{n-1} \to \mathbb{R}P^n$ as the attaching map. It follows by induction on n that $\mathbb{R}P^n$ has a cell complex structure $e^0 \cup e^1 \cup \cdots \cup e^n$ with one cell e_i in each dimension $i \leq n$.

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- The complex projective plane $\mathbb{C}P^2$ has a simpler cell complex structure, consisting of starting with a single point $X^0 = \{(0:0:1)\}$, and then attaching a 2-cell (a copy of $\mathbb{C} = \mathbb{R}^2$) like a balloon to form X^2. A copy of $\mathbb{C}^2 = \mathbb{R}^4$ is then attached to form X^4.

Cellular Homology

Definition

For a CW complex $X = X^0 \subset \cdots \subset X^n$ define $C_k = Z^{\# k\text{-cells}}$ to be the free abelian group generated by the k-cells $B_\alpha^{(k)} = \left(D_\alpha^{(k)}\right)^{\circ}$.

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Definition

The **cellular boundary map** $d_{k+1} : C_{k+1} \to C_k$ *is* $d_{k+1}(B_\alpha^{(k+1)}) = \sum_\beta deg_{\alpha\beta} \cdot B_\alpha^{(k)}$, *where* $deg_{\alpha\beta}$ *is the degree of the composite map* $\partial \overline{B_\alpha^{(k+1)}} \to X^k \to \overline{B_\beta^{(k)}}$.

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- **Remark** The first map above is the cellular attaching map from the boundary of the closure of the ball $B_\alpha^{(k+1)}$ to the k-skeleton, and the second map is the quotient map formed by collapsing $X^k \setminus B_\beta^{(k)}$ to a point. The composite is a map from a k-sphere to another k-sphere, which has a **degree**.

Cellular Homology (contd.)

- It is known that the cellular boundary maps make the groups C_k into a **chain complex**, which is the following sequence of maps:

$$0 \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \longrightarrow 0$$

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Definition

The **cellular homology groups** *of a X are the abelian groups defined through the cellular chain complex above as such:*

$$H_i(X) = \frac{ker(d_i)}{im(d_{i+1})}$$

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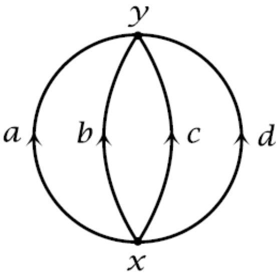

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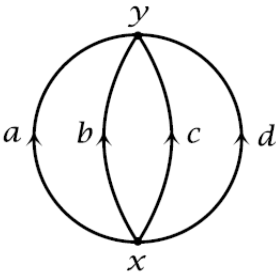

- Consider the space above, X.
- We will consider how the homology groups look exactly in this case, before and after attaching a 2-cell.

Cellular Homology Examples (contd.)

- Since $\mathbb{C}P^2$ consists of a point, a 2-cell, and a 4-cell, its cellular chain complex becomes:

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- The first map here is multiplication by 2 and the second is the zero map, from the degrees of the attaching maps.

Cellular Homology Examples (contd.)

- Since $\mathbb{C}P^2$ consists of a point, a 2-cell, and a 4-cell, its cellular chain complex becomes:

$$\cdots \to 0 \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to 0$$

- The homology groups are $H_0 = H_2 = H_4 = \mathbb{Z}$, $H_1 = H_3 = 0$.
- For $\mathbb{R}P^2$, the chain complex looks like:

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$$

- The first map here is multiplication by 2 and the second is the zero map, from the degrees of the attaching maps.
- It follows that $H_2 = 0$, $H_1 = \mathbb{Z}/2\mathbb{Z}$ and $H_0 = 0$

Cellular Cohomology

- We can now define the cellular cohomology by dualizing the chain complex above.

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- The coboundary maps form a **cochain complex**.

Definition

The i-th **cellular cohomology group** is

$$H^i(X) = ker(d_{i+1}^*)/im(d_i^*)$$

Cellular Cohomology Examples

- The cellular cochain complex for $\mathbb{C}P^2$ is

$$0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to 0 \to 0 \to \cdots$$

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- There is in fact additional structure on the cohomology groups (it is a contravariant functor).
- The direct sum of the cohomology groups

$$H^*(X) = \bigoplus H^i(X)$$

has a ring structure given by the **cup** product, which is the dual of the "cap" product on homology groups.

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- The Cohomology Ring

4 Intersections of the Schubert Varieties
- Schubert Cells
- Important Theorems

What are the Schubert Cells?

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- The subset of the Grassmannian whose points have a particular reduced row echelon form constitutes a **Schubert Cell**.
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- We can enumerate the number of Schubert cells using partitions and young diagrams.
- This is equivalent to the coming definition.

What are Schubert Cells? (contd.)

The **Schubert cell** is defined as $\Omega^o_\lambda =$
$\{V \in \mathrm{Gr}(k,n) \mid \dim(V \cap \langle e_1, \ldots, e_r \rangle) = i \text{ for } a - \lambda_i \leq r \leq a - \lambda_{i+1}\}$,
where $a = n - k + i$.

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- We can now count dimension using this construction.
- Each $*$ can be a complex number, so we have $dim(\Omega_\lambda^o) = k(n-k) - |\lambda|$. and $dim(Gr(k,n)) = k(n-k)$.

Schubert Varieties

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- Flags are very important in Schubert Calculus and have their own structure.
- Certain flags, e.g. transverse flags, standard and opposite flags, are enough for our computations.

Big Theorem 1

Theorem

The cohomology ring $H^*(\mathrm{Gr}(k,n))$ *has a* $\mathbb{Z}$*-basis given by the classes*

$$\sigma_\lambda := [\Omega_\lambda(F_\bullet)] \in H^{2|\lambda|}(\mathrm{Gr}(n,k))$$

for λ *a partition fitting inside the ambient rectangle. The cohomology* $H^*(\mathrm{Gr}(k,n))$ *is a graded ring, so* $\sigma_\lambda \cdot \sigma_\mu \in H^{2|\lambda|+2|\mu|}(\mathrm{Gr}(k,n))$*, and we have*

$$\sigma^\lambda \cdot \sigma^\mu = [\Omega_\lambda(F_\bullet) \cap \Omega_\mu(E_\bullet)]$$

where $F_\bullet$ *and* $E_\bullet$ *are the standard and opposite flags.*

Big Theorem 2

Theorem

There is a ring isomorphism

$$H^*(G(k,n)) \cong \Lambda(x_1, x_2, \ldots)/(s_\lambda | \lambda \not\subset B)$$

where B is the ambient rectangle and $(s_\lambda | \lambda \not\subset B)$ is the ideal generated by the Schur functions. The isomorphism sends the Schubert class σ_λ to the Schur function s_λ.

Thank You

Quesions?