The Borsuk-Ulam Theorem

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1 Introduction

The Borsuk-Ulam Theorem is a very well-known theorem in Topology. It has been studied extensively, and has a plethora of applications in combinatorics and topology. We will see a classic construction of the theorem and its proof, then look at some generalizations and other forms of proof directions, as well as some mirror theorems in combinatorics. We assume knowledge of basic homotopy and the theory surrounding the fundamental group.

2 The Classic Approach

Definition 1. *If x is a point of S^n, then its antipode is the point $-x$. A map $h : S^n \to S^n$ is said to be antipode-preserving if $h(-x) = -h(x)$ for all $x \in S^n$.*

Theorem 2.1. *If $h : S^1 \to S^1$ is continuous and antipode-preserving, then h is not nulhomotopic.*

Proof. Let b_0 be the point $(1, 0)$ of S^1. Let $\rho : S^1 \to S^1$ be a rotation of S^1 that maps $h(b_0)$ to b_0. Since ρ preserves antipodes, so does the composite $\rho \circ h$. Furthermore, if H were a homotopy between h and a constant map, then $\rho \circ H$ would be a homotopy between $\rho \circ h$ and a constant map. Therefore, it suffices to prove the theorem under the additional hypothesis that $h(b_0) = b_0$.

Step 1. Let $q : S^1 \to S^1$ be the map $q(z) = z^2$, where z is a complex number. Or in real coordinates, $q(\cos\theta, \sin\theta) = (\cos 2\theta, \sin 2\theta)$. The map q is a quotient map, being continuous, closed, and surjective. The inverse image under q of any point of S^1 consists of two antipodal points z and $-z$ of S^1. Because $h(-z) = -h(z)$, one has the equation $q(h(-z)) = q(h(z))$. Therefore, because q is a quotient map, the map $q \circ h$ induces a continuous map $k : S^1 \to S^1$ such that $k \circ q = q \circ h$.

$$\begin{array}{ccc} S^1 & \xrightarrow{h} & S^1 \\ \downarrow q & & \downarrow q \\ S^1 & \xrightarrow{k} & S^1 \end{array}$$

Note that $q(b_0) = h(b_0) = b_0$, so that $k(b_0) = b_0$ as well. Also, $h(-b_0) = -b_0$.

Step 2. We show that the homomorphism k_* of $\pi_1(S^1, b_0)$ with itself is nontrivial.

For this purpose, we first show that q is a covering map. The proof is similar to the proof that the standard map $p : \mathbb{R} \to S^1$ is a covering map. If, for instance, U is the subset of S^1 consisting of those points having positive second coordinate, then $p^{-1}(U)$ consists of those points of S^1 lying in the first and third quadrants of $\mathbb{R}^2$. The map q carries each of these sets homeomorphically onto U. Similar arguments apply when U is the intersection of S^1 with the open lower half-plane, or with the open right and left half-planes.

Second, we note that if $\tilde{f}$ is any path in S^1 from b_0 to $-b_0$, then the loop $f = q \circ \tilde{f}$ represents a nontrivial element of $\pi_1(S^1, b_0)$. For $\tilde{f}$ is a lifting of f to S^1 that begins at b_0 and does not end at b_0.

Finally, we show k_* is nontrivial. Let $\tilde{f}$ be a path in S^1 from b_0 to $-b_0$, and let f be the loop $q \circ \tilde{f}$. Then $k_*[f]$ is not trivial, for $k_*[f] = [k \circ (q \circ \tilde{f})] = [q \circ (h \circ \tilde{f})]$; the latter is nontrivial because $h \circ \tilde{f}$ is a path in S^1 from b_0 to $-b_0$.

Step 3. Finally, we show that the homomorphism h_* is nontrivial, so that h cannot be nulhomotopic.

The homomorphism k_* is injective, being a nontrivial homomorphism of an infinite cyclic group with itself. The homomorphism q_* is also injective; indeed, q_* corresponds to multiplication by two in the group of integers. It follows that $k_* \circ q_*$ is injective. Since $q_* \circ h_* = k_* \circ q_*$, the homomorphism h_* must be injective as well. □

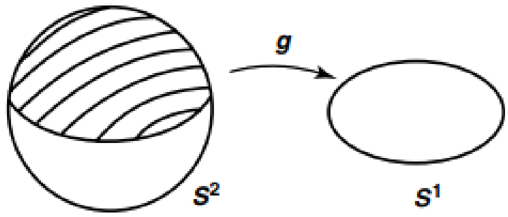

Figure 1: The continuous extension

The next theorem will be used directly to prove the main result.

Theorem 2.2. *There is no continuous antipode-preserving map $g : S^2 \to S^1$.*

Proof. Suppose $g : S^2 \to S^1$ is continuous and antipode preserving. Let us take S^1 to be the equator of S^2. Then the restriction of g to S^1 is a continuous antipode-preserving map h of S^1 to itself. By the preceding theorem, h is not nulhomotopic. But the upper hemisphere E of S^2 is homeomorphic to the ball B^2, and g is a continuous extension of h to E. This is not possible. $\square$

We are now ready to state and prove the main theorem.

Theorem 2.3 (Borsuk-Ulam theorem (for S^2)). *Given a continuous map $f : S^2 \to \mathbb{R}^2$, there is a point x of S^2 such that $f(x) = f(-x)$.*

Proof. Suppose that $f(x) \neq f(-x)$ for all $x \in S^2$. Then the map

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

is a continuous map $g : S^2 \to S^1$ such that $g(-x) = -g(x)$ for all x. $\square$

We can now also state the general version, and other equivalent formulations of the Borsuk-Ulam theorem.

Theorem 2.4 (Versions of Borsuk-Ulam). *The following are equivalent:*

1. ***(Generalized)*** *If $f : S^n \to \mathbb{R}^n$ is continuous then there exists $x \in S^n$ such that $f(-x) = f(x)$.*
2. *If $f : S^n \to \mathbb{R}^n$ is antipodal, i.e. $f(-x) = -f(x)$, then there exists $x \in S^n$ such that $f(x) = 0$.*
3. *There is no antipodal map $S^n \to S^{n-1}$.*
4. *If $\{A_1, A_2, \ldots, A_{n+1}\}$ is a closed cover of S^n, then some A_i contains a pair of antipodal points.*
5. *Generalizing (4), each A_i is either open or closed.*

3 The Combinatorial Approach

The combinatorial version of the Borsuk-Ulam is known as the Tucker's lemma. One of the most famous applications of Borsuk-Ulam is the Brouwer fixed-point theorem, and indeed, there is also a combinatorial analog of this implication. This is called the Sperner's lemma. We will state Tucker's lemma here, without proof.

Theorem 3.1 (Tucker's lemma). *Consider a triangulation of B^n with vertices labeled $\pm 1, \pm 2, \ldots, \pm n$, such that the labeling is antipodal on the boundary. Then there exists an edge (1-simplex) whose endpoints have opposite labels $i, -i$.*

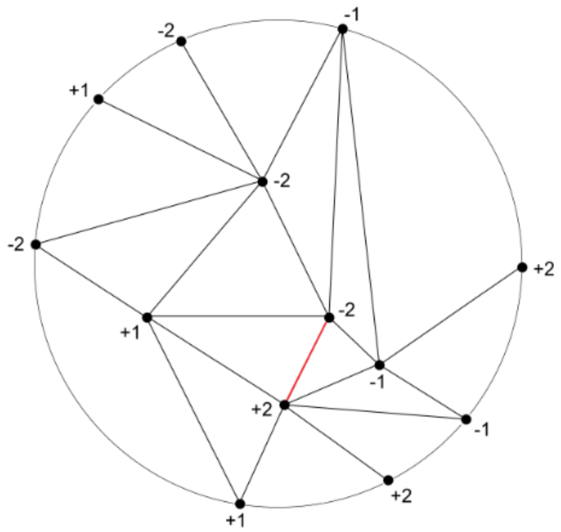

Figure 2: An example triangulation for Tucker's lemma, with n=2

Sperner's lemma has to do with the coloring of a triangulation like the one shown above.

4 Applications

We state here a few applications of the Borsuk-Ulam theorem, skipping details of the proofs. These are readily available.

Theorem 4.1 (**Ham-Sandwich Theorem**). *Let $A_1, A_2, \ldots, A_n$ be n measurable subsets in $\mathbb{R}^n$. Then there exists a hyperplane H in $\mathbb{R}^n$ such that H simultaneously bisects the n objects. In other words, for each i, the hyperplane H divides A_i into two parts of equal measure.*

Proof for $\mathbb{R}^2$. This is called the **bisection theorem**. We take two bounded polygonal regions A_1 and A_2 in the plane $\mathbb{R}^2 \times 1$ in $\mathbb{R}^3$, and show there is a line L in this plane that bisects each of them.

Given a point u of S^2, let us consider the plane P in $\mathbb{R}^3$ passing through the origin that has u as its unit normal vector. This plane divides $\mathbb{R}^3$ into two half-spaces; let $f_i(u)$ equal the area of that portion of A_i that lies on the same side of P as does the vector u.

If u is the unit vector $\mathbf{k}$, then $f_i(u) =$ area A_i; and if $u = -\mathbf{k}$, then $f_i(u) = 0$. Otherwise, the plane P intersects the plane $\mathbb{R}^2 \times 1$ in a line L that splits $\mathbb{R}^2 \times 1$ into two half-planes, and $f_i(u)$ is the area of that part of A_i that lies on one side of this line.

Replacing u by $-u$ gives us the same plane P, but the other half-space, so that $f_i(-u)$ is the area of that part of A_i that lies on the other side of P from u. It follows that

$$f_i(u) + f_i(-u) = \text{area } A_i.$$

Now consider the map $F : S^2 \to \mathbb{R}^2$ given by $F(u) = (f_1(u), f_2(u))$. The Borsuk-Ulam theorem gives us a point u of S^2 for which $F(u) = F(-u)$. Then $f_i(u) = f_i(-u)$ for $i = 1, 2$, that $f_i(u) = \frac{1}{2}$area A_i, as desired. □

This can be used to solve the infamous Necklace splitting problem!

Theorem 4.2 (**Brouwer Fixed-Point theorem**). *Let D^n be the closed unit disk in $\mathbb{R}^n$. Any continuous function $f : D^n \to D^n$ has at least one fixed point. In other words, there exists a point $x \in D^n$ such that $f(x) = x$.*

This theorem has many proofs, and also has extensive applications in many areas of Mathematics. The final application that we will see is called Tverberg's theorem - another theorem from combinatorics. The Tverberg's theorem and its applications is an active area of research. Th

Theorem 4.3 (**Tverberg's theorem**). *Let $n \geq 1, r \geq 2$.*
Every set of $nr + r - n$ points in $\mathbb{R}^n$ can be partitioned as $A_1 \cup A_2 \cup \ldots \cup A_r$, such that

$$conv(A_1) \cap conv(A_2) \cap \ldots \cap conv(A_r) \neq \emptyset.$$

Note that the operator $conv(\cdot)$ represents the convex hull. The theorem states that there exists a way to use the points to create convex hulls such that they all intersect in at least one point.