Schubert Calculus and the Cohomology of the Grassmannian

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The Cohomology of the Grassmannian

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Abstract

While counting is the simplest mathematical exercise, it can become notoriously involved when we try to enumerate geometric objects and in particular their intersections; this is precisely the subject of the 15th Hilbert Problem (of the famous 23 published in 1900), namely the formalization of Schubert Calculus. We will develop remarkable connections between Algebraic Geometry, Topology and Combinatorics through the enumerative tools of this subject. There is a surprising translation between structures on topological invariants (Cohomology Groups) and structures of purely combinatorial objects (Young Tableaux and Symmetric Polynomials) of a very important class of geometric objects: a differentiable manifold known as the Grassmannian. These translations rely on very powerful result in Topology, namely Poincaré Duality. We will develop necessary general preliminaries (e.g. an Introduction to Cohomology, the Calculus of Tableaux, structures on the Grassmannian) and then consider these translations in particular. Applications of Schubert Calculus and its ideas range from Intersection theory, to even purely geometric questions, e.g. the Toeplitz Square Peg problem, but the interesting nature of these unlikely mathematical connections warrants attention in its own right.

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Chapter 1

Introduction and Classical Geometry

1.1 Counting

The most rudimentary problem of Mathematics is counting things. In essence the goal of this project is also a type of enumeration. What we want to enumerate is intersections of geometric objects; consider the intersection of two lines in $\mathbb{R}^2$, a simple problem of Linear Algebra.

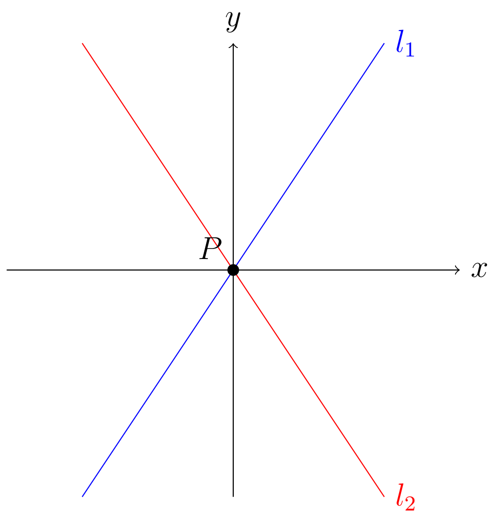


Figure 1.1: Two one-dimensional subspaces of $\mathbb{R}^2$

Our aim is to solve *similar* linear intersection problems in Euclidean space ($\mathbb{C}$ or $\mathbb{R}$). We will be dealing with $\mathbb{C}$ primarily, as we will in many cases interpret our problems as solutions to polynomial equations, and we want to work over an algebraically closed field, so we can see all of our solutions. Let's consider a series of intersection problems.

First, consider the question: how many points lie on two distinct lines. The answer is straightforwardly 1, and the degenerate case of no intersections can be appropriately handled by working in projective space. The next, slightly harder question we consider is show many lines pass through four given lines in three-dimensional space. In general, the answer will turn out to be 2. This is a very illustrative problem that

Herman Schubert solved, and we will return to it with a more sophisticated approach.

We can ask an even harder intersection question: how many twisted cubics are tangential to twelve given quadrics in three-dimensional space. The definitions of these objects will become clear soon when we define algebraic curves. The twisted cubic is parameterized as the curve (t, t^2, t^3) and the condition of being tangential is simply an intersection with multiplicity 2. The answer is in order of trillions: $5, 819, 539, 783, 680$. This is a significant increase in complexity and a combinatorial explosion of possible intersections, and the authors David Eisenbud and Joe Harris compared the fact that Herman Schubert was able to get to this answer in 1879 without very complicated mathematical machinery to landing a jumbo-jet blindfolded!

Finally, we can also pose more abstract questions: how many k-dimensional subspaces of $\mathbb{C}^n$ intersect each of the $k(n-k)$ fixed subspaces of dimension $(n-k)$ non trivially. We will require a great deal of theory to deal with this question, and in fact, the 15^{th} problem from the famous 23 problems posed by David Hilbert at the turn of the century in 1900 was to formalize Schubert's arguments, since many of his proofs relied on sketchy continuity arguments, generality and more blunt tools like dimension counting and a careless application of Bezout's theorem. This naive application usually led to wrong answers, and algebraic geometers introduced vague "fudge factors" to deal with the mismatch. These interventions seemed arbitrary and problem-specific, and therefore was the subject of one of Hilbert's problems.

Intersection theory, enumerative and arithmetic geometry, and incidence combinatorics are on a vast and rigorous foundation today, and Schubert Calculus is one of the more refined tools that we use. In this exposition, we will first introduce some classical geometry problems and theorems, and then the prerequisite theory needed to understand Schubert Calculus: the Grassmannian from Algebraic Geometry, Young Tableaux and Symmetric Functions from Combinatorics, and Cohomology from Algebraic Topology. The introductions are short, but meant to be sufficiently complete. Eventually, we will endeavour to construct the 'first principles' of Schubert Calculus.

1.2 Algebraic Plane Curves

In this section, we will begin building up some theory to prove some classical theorems in Euclidean geometry. The main goal is to prove the Cayley-Bacharach Theorem (also known as the 8-point theorem of cubic curves); we will then be able to prove the famous Pappus' Theorem and Pascal's Theorem using Cayley-Bacharach. [7]

Definition 1. *An algebraic (affine) plane curve of degree d over some field k is a curve γ of the form*

$$\gamma = \{(x, y) \in k^2 : P(x, y) = 0\}$$

where P is a non-constant polynomial of degree d.

Examples include

- Degree 1 (linear) curves $\{(x, y) \in k^2 : ax + by = c\}$, which are lines in k^2;

- Degree 2 (quadric) curves $\{(x, y) \in k^2 : ax^2 + bxy + cy^2 + dx + ey + f = 0\}$, which include classical conic sections (circles, hyperbolae, parabolae, ellipses) when $k = \mathbb{R}$, and also the reducible case of a union of two lines;

- Degree 3 (cubic) curves $\{(x, y) \in k^2 : ax^3+bx^2y+cxy^2+dy^3+ex^2+fxy+gy^2+hx+iy+j = 0\}$, which include the elliptic curves $\{(x, y) \in k^2 : y^2 = x^3+ax+b\}$ (with non-zero discriminant $\Delta := -16(4a^3+27b^2)$, so that the curve is smooth) as examples (ignoring some technicalities when k has characteristic two or three), but also include the reducible examples of the union of a line and a conic section, or the union of three lines.

Algebraic affine plane curves can also be extended to the projective plane $\mathbb{P}^2_k = \{[x, y, z] : (x, y, z) \in k^3 \backslash 0\}$ by homogenising the polynomial. For instance, the affine quadric curve $\mathbb{P}k^2 = \{[x : y : z] : (x, y, z) \in k^3 \backslash \{0\}\}$ would become $\{[x : y : z] \in \mathbb{P}k^2 : ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0\}$. Refer to Chapter 2 for details.

We will state, without proof, a famous theorem in algebra and geometry. This will be used repeatedly to prove the Cayley-Bacharach Theorem.

Theorem 1.2.1 (Bezout's theorem). *If a degree d curve γ and a degree d' curve γ' have no common component, then they intersect in at most dd' points (and if the underlying field k is algebraically closed, one works projectively, and one counts intersections with multiplicity, they intersect in exactly dd' points).*

Note: The common components in the aforementioned theorem refer to the reducible components of an algebraic curve.

Thus, for instance, two distinct lines intersect in at most one point; a line and a conic section intersect in at most two points; two distinct conic sections intersect in at most four points; a line and an elliptic curve intersect in at most three points; two distinct elliptic curves intersect in at most nine points; and so forth.

Using facts from Linear Algebra, we can easily build algebraic curves by considering general plane equations and applying constraints to these equations by specifying a particular number of points in the plane through which the curve must pass. For example:

- For any two points A_1, A_2 one can find a line $\{(x, y) : ax + by = c\}$ passing through the points A_1, A_2, because this imposes two linear constraints on three unknowns a, b, c and is thus guaranteed to have at least one solution;

- Given any five points $A_1, \ldots, A_5$, one can find a quadric curve passing through these five points (though note that if three of these points are collinear, then this curve cannot be a conic thanks to Bezout's theorem as the line passing through these points and the curve can only intersect in at most two points, and is thus necessarily reducible to the union of two lines);

- Given any nine points $A_1, \ldots, A_9$, one can find a cubic curve going through these nine points.

Next, we must deal with the uniqueness of algebraic curves that are built through these *polynomial methods.* This will be relevant to our discussion of the Cayley-Bacharach theorem.

In the linear case, it is clear that the line determined by two distinct points is always unique. The higher degree cases are more complicated. For instance, five collinear points do not determine a unique quadric curve; one can simply take the union of the line containing those five points, and any other arbitrary line; every choice of the other line determines a different quadric. Similarly, eight points on a conic section plus one additional point determine more than one cubic curve, as one can take that conic section plus an arbitrary line going through the additional point.

This issue of uniqueness is resolved if some "general" situation is imposed on these points. For instance, given five points, no three of which are collinear, there can be at most one quadric curve that passes through these points (because these five points cannot lie on the union of two lines, and by Bézout's theorem they cannot simultaneously lie on two distinct conic sections). Here, the general hypothesis is that no three points lie on a hypersurface (a line in this case).

For cubic curves, the situation is more complicated still. Consider for instance two distinct cubic curves $\gamma_0 = \{P_0(x, y) = 0\}$ and $\gamma_\infty = \{P_\infty(x, y) = 0\}$ that intersect in precisely nine points $A_1, \ldots, A_9$ (note from Bézout's theorem that this is an entirely typical situation). Then there is in fact an entire one-parameter family of cubic curves that pass through these points, namely the curves $\gamma_t = \{P_0(x, y) + tP_\infty(x, y) = 0\}$ or any t$\in k \cup \{\infty\}\}$. This leads immediately to the theorem in the next section.

1.3 Cayley-Bacharach Theorem

Theorem 1.3.1 (Cayley-Bacharach theorem). *Let $\gamma_0 = \{P_0(x, y) = 0\}$ and $\gamma_\infty = \{P_\infty(x, y) = 0\}$ be two cubic curves that intersect (over some algebraically closed field k) in precisely nine distinct points $A_1, \ldots, A_9 \in k^2$. Let P be a cubic polynomial that vanishes on eight of these points (say $A_1, \ldots, A_8$). Then P is a linear combination of P_0, P_∞, and in particular vanishes on the ninth point A_9.*

Proof. (by Husemöller) We assume for contradiction that there is a cubic polynomial P that vanishes on $A_1, \ldots, A_8$, but is not a linear combination of P_0 and P_∞.

We first make some observations on the points $A_1, \ldots, A_9$. No four of these points can be collinear, because then by Bézout's theorem, P_0 and P_∞ would both have to vanish on this line, contradicting the fact that γ_0, γ_∞ meet in at most nine points. For similar reasons, no seven of these points can lie on a quadric curve.

One consequence of this is that any five of the $A_1, \ldots, A_9$ determine a unique quadric curve σ. The existence of the curve follows from linear algebra as discussed previously. If five of the points lie on two different quadric curves σ, σ', then by Bezout's theorem, they must share a common line; but this line can contain at most three of the five points, and the other two points determine uniquely the other line that is the component of both σ and σ', and the claim follows.

Now suppose that three of the first eight points, say A_1, A_2, A_3, are collinear, lying on a line ℓ. The remaining five points $A_4, \ldots, A_8$ do not lie on ℓ, and determine a unique quadric curve σ by the previous discussion. Let B be another point on ℓ, and let C be a point that does not lie on either ℓ or σ. By linear algebra, one can find a non-trivial linear combination $Q = aP + bP_0 + cP_\infty$ of P, P_0, P_∞ that vanishes at both B and C. Then Q is a cubic polynomial that vanishes on the four collinear points A_1, A_2, A_3, B and thus vanishes on ℓ, thus the cubic curve defined by Q consists of ℓ and a quadric curve. This curve passes through $A_4, \ldots, A_8$ and thus equals σ. But then C does not lie on either ℓ or σ despite being a vanishing point of Q, a contradiction. Thus, no three points from $A_1, \dots, A_8$ are collinear.

In a similar vein, suppose next that six of the first eight points, say $A_1, \dots, A_6$, lie on a quadric curve σ; as no three points are collinear, this quadric curve cannot be the union of two lines, and is thus a conic section. The remaining two points A_7, A_8 determine a unique line $\ell = \overleftrightarrow{A_7A_8}$. Let B be another point on σ, and let C be another point that does not lie on either ℓ and σ. As before, we can find a non-trivial cubic $Q = aP + bP_0 + cP_\infty$ that vanishes at both B, C. As Q vanishes at seven points of a conic section σ, it must vanish on all of σ, and so the cubic curve defined by Q is the union of σ and a line that passes through A_7 and A_8, which must necessarily be ℓ. But then this curve does not pass through C, a contradiction. Thus no six points in $A_1, \dots, A_8$ lie on a quadric curve.

Finally, let ℓ be the line through the two points A_1, A_2, and σ the quadric curve through the five points $A_3, \dots, A_7$; as before, σ must be a conic section, and by the preceding paragraphs we see that A_8 does not lie on either σ or ℓ. We pick two more points B, C lying on ℓ but not on σ. As before, we can find a non-trivial cubic $Q = aP + bP_0 + cP_\infty$ that vanishes on B, C; it vanishes on four points on ℓ and thus Q defines a cubic curve that consists of ℓ and a quadric curve. The quadric curve passes through $A_3, \dots, A_7$ and is thus σ; but then the curve does not pass through A_8, a contradiction. This contradiction finishes the proof of the proposition. □

1.4 Pappus' and Pascal's Theorem

Theorem 1.4.1 (Pappus' theorem). *Let ℓ, ℓ' be two distinct lines, let A_1, A_2, A_3 be distinct points on ℓ that do not lie on ℓ', and let B_1, B_2, B_3 be distinct points on ℓ' that do not lie on ℓ. Suppose that for $ij = 12, 23, 31$, the lines $\overleftrightarrow{A_iB_j}$ and $\overleftrightarrow{A_jB_i}$ meet at a point C_{ij}. Then the points C_{12}, C_{23}, C_{31} are collinear.*

Proof. We may assume that C_{12}, C_{23} are distinct, since the claim is trivial otherwise. Let γ_0 be the union of the three lines $\overleftrightarrow{A_1B_2}$, $\overleftrightarrow{A_2B_3}$, and $\overleftrightarrow{A_3B_1}$ (the purple lines in Figure 2), let γ_1 be the union of the three lines $\overleftrightarrow{A_2B_1}$, $\overleftrightarrow{A_3B_2}$, and $\overleftrightarrow{A_1B_3}$ (the dark blue lines), and let γ be the union of the three lines ℓ, ℓ', and $\overleftrightarrow{C_{12}C_{23}}$ (the other three lines). By construction, γ_0 and γ_1 are cubic curves with no common component that meet at the nine points $A_1, A_2, A_3, B_1, B_2, B_3, C_{12}, C_{23}, C_{31}$. Also, γ is a cubic curve that passes through the first eight of these points, and thus also passes through the ninth point C_{31}, by the Cayley-Bacharach theorem. The claim follows (note that C_{31} cannot lie on ℓ or ℓ'). □

Theorem 1.4.2 (Pascal's theorem). *Let $A_1, A_2, A_3, B_1, B_2, B_3$ be distinct points on a conic section σ. Suppose that for $ij = 12, 23, 31$, the lines $\overleftrightarrow{A_iB_j}$ and $\overleftrightarrow{A_jB_i}$ meet at a point C_{ij}. Then the points C_{12}, C_{23}, C_{31} are collinear.*

Proof. Repeat the proof of Pappus' theorem, with σ taking the place of $\ell \cup \ell'$. (Note that as any line meets σ in at most two points, the C_{ij} cannot lie on σ.) □

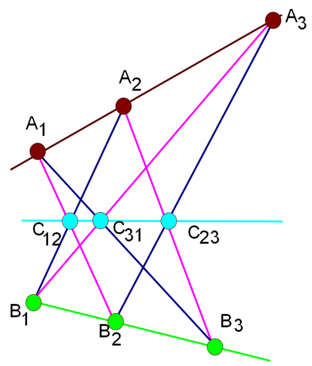


Figure 1.2: Pappus' Theorem

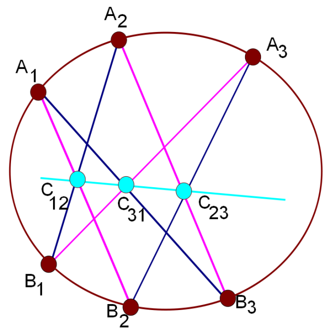


Figure 1.3: Pascal's Theorem

One can also view Pappus's theorem as the degenerate case of Pascal's theorem, when the conic section degenerates to the union of two lines.

1.5 Sylvester-Gallai Theorem

The last 'classical' theorem I want to introduce has a very elegant proof. The incidence combinatorics where the theory from this theorem leads is vast, and has also seen progress in the last decade, for example by Terence Tao.

Definition 2. *Give a set S of a finite number of points in the plane, a* ***k-rich line*** *of S is a line that passes through exactly k points. When $k = 2$, we say that the line is an ordinary line.*

Theorem 1.5.1 (Sylvester-Gallai theorem). *Let S be a set of points in $\mathbb{R}^2$, not all collinear. Then there exists at least one ordinary line of S.*

Proof. (By Kelly) Consider a finite set S of points that are not all collinear. Define a connecting line as one that passes through at least two points in the collection. As S is finite, there must exist a point P and a connecting line ℓ such that P and ℓ are a positive distance apart but closer than any other point-line pair.

Assume that ℓ is not ordinary, since otherwise we are done. In that case, ℓ passes through at least three points of S. At least two of these points must lie on the same side of P', the perpendicular projection of P on ℓ. Let these points be B and C, with B being the closest to P'. Draw the connecting line m through P and C, and the perpendicular from B to B' on m. Consequently, BB' is shorter than PP', as $PP'C$ and $BB'C$ are similar triangles, one nested within the other (See figure below).

However, this conclusion contradicts the original definition of P and ℓ as the point-line pair with the smallest positive distance. Therefore, the assumption that ℓ is not ordinary must be false. Hence, ℓ is ordinary. □

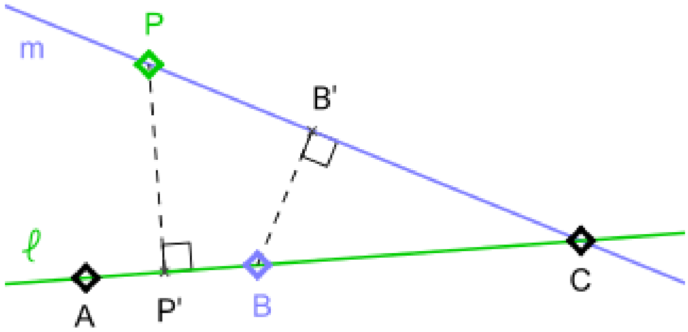


Figure 1.4: Proof of the Sylvester-Gallai theorem

Chapter 2

Geometry of the Grassmannian

In this chapter we discuss basic notions Algebraic Geometry. We first discuss the geometry of affine and projective spaces, and then introduce the Grassmannian as a projective variety.

2.1 Affine Spaces and Varieties

An affine space is a generalization of usual Euclidean spaces, such that the 'vector space' notions of distance metrics and angles are discarded, leaving only a certain geometric structure. For all intents and purposes, these may be treated as Euclidean spaces. The definition is as follows.

Definition 3. *Affine Space Given a field k and a positive integer n, the set*

$$\mathbb{A}^n = k^n = \{(a_1, \ldots, a_n) | a_1, \ldots, a_n \in k\}$$

is the affine space over k of dimension n.

When $k = \mathbb{R}$ or $k = \mathbb{C}$, we get the usual spaces that we are familiar with. If $n = 1$, we call $\mathbb{A}^1$ the affine line, and if $n = 2$, we call $\mathbb{A}^2$ the affine plane.

The primary objects of focus in Algebraic Geometry are known as varieties, which are the common zero loci of polynomials. First, we must define the ring of polynomials in some variables.

Definition 4 (Ring of Polynomials)**.** *Let R be a commutative ring with unity. The **ring of polynomials** in n variables $x_1, x_2, \ldots, x_n$ over R, denoted by $R[x_1, x_2, \ldots, x_n]$, consists of finite sums of terms $c_\alpha x^\alpha$, where $c_\alpha \in R$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ with non-negative integers α_i. Addition and multiplication are defined by:*

$$f + g = \sum_\alpha (c_\alpha + d_\alpha) x^\alpha \quad \textit{and} \quad f \cdot g = \sum_\gamma \left(\sum_{\alpha+\beta=\gamma} c_\alpha d_\beta \right) x^\gamma.$$

We are now in a position to describe algebraic (affine) varieties as the zero locus of polynomials. It will turn out that an analog, called projective varieties, is much more relevant to our discussion of solving intersection problems. In general, we want to 'projectivize' our discussion.

Definition 5 (Affine Variety)**.** *An **affine variety** over a field k is a subset of affine n-space k^n that is the common zero locus of a set of polynomials in $k[x_1, x_2, \ldots, x_n]$.*

Specifically, if $S \subseteq k[x_1, x_2, \ldots, x_n]$ is a set of polynomials, the affine variety $V(S)$ is defined as:

$$V(S) = \{\mathbf{a} \in k^n \mid f(\mathbf{a}) = 0 \text{ for all } f \in S\}.$$

*If S can be taken to be finite, say $S = \{f_1, f_2, \ldots, f_m\}$, then $V(S)$ is called an **algebraic set**, and we write:*

$$V(f_1, f_2, \ldots, f_m) = \{\mathbf{a} \in k^n \mid f_i(\mathbf{a}) = 0 \text{ for all } i = 1, 2, \ldots, m\}.$$

Example: Consider the affine varieties defined by the following polynomials in $\mathbb{R}[x, y]$:

- $f_1(x, y) = x + y - 1$, which represents a line.
- $f_2(x, y) = x^2 + y^2 - 1$, which represents a circle of radius 1 centered at the origin.

The affine variety $V(f_1, f_2)$ is the set of points (x, y) in $\mathbb{R}^2$ that simultaneously satisfy both equations, which are $(0, 1)$ and $(1, 0)$.

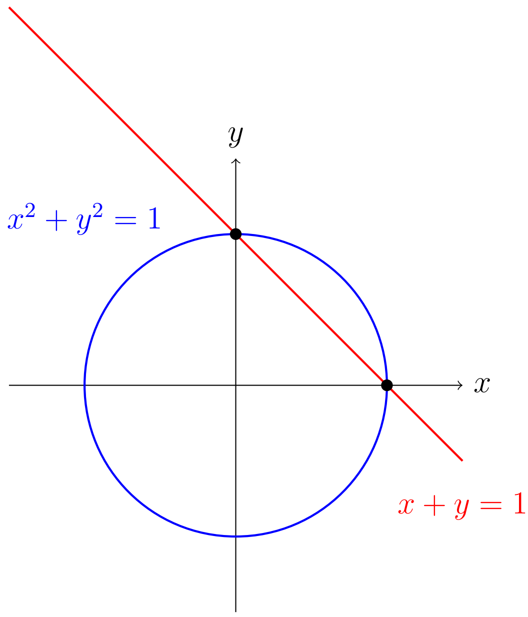


Figure 2.1: Intersection of Two Varieties is a Variety

As it turns out, the affine varieties can be considered as closed sets of the **Zariski topology**. You can also show that affine geometry is completely equivalent to corresponding notions in commutative algebra, specifically the ring of polynomials and subsets called ideals of a ring - there is a contravariant functor associated with this translation, denoted by $A(\cdot)$, that gives the algebraic functions on a given space.

2.2 Projective Geometry

Definition 6. *The n-dimensional **projective space** $\mathbb{P}^n_k$ over a field k is the set of equivalence classes in $k^{n+1} \setminus \{(0, 0, \ldots, 0)\}$ with respect to the relation $\sim$ given by scalar multiplication, that is,*

$$(x_0, x_1, \ldots, x_n) \sim (y_0, y_1, \ldots, y_n)$$

if and only if there exists $a \in k \setminus \{0\}$ such that $ax_i = y_i$ for all i. We write $(x_0 : x_1 : \cdots : x_n)$ for the equivalence class in $\mathbb{P}^n_k$ containing $(x_0, \ldots, x_n)$, and we refer to $(x_0 : x_1 : \cdots : x_n)$ as a point *in $\mathbb{P}^n_k$ in* homogeneous coordinates.

Note that a *point* in $\mathbb{P}^n_k$ is a line through the origin in k^{n+1}. In particular, a line through the origin consists of all scalar multiples of a given nonzero vector.

A common example of projective space is the **Riemann sphere** $\mathbb{P}^1_{\mathbb{C}}$ for example, which is the one-point compactification of the complex plane.

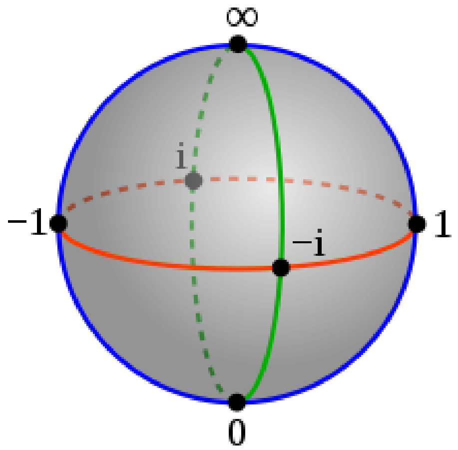


Figure 2.2: Riemann Sphere

This antipodal identification will be an important way to think about $\mathbb{P}^n$. This can be generalized to have an n-fold identification space called the *Dunce cap*. The projective spaces have their own geometry through a group of projective transformations, which are simply maps $f : \mathbb{P}^n \to \mathbb{P}^n$ of the form

$$f(x_0 : x_1 : \cdots : x_n) = (y_0 : y_1 : \cdots : y_n)$$

where for each i,

$$y_i = a_{i0}x_0 + a_{i1}x_1 + \cdots + a_{in}x_n$$

for some fixed constants $a_{ij} \in \mathbb{C}$ such that the $(n+1) \times (n+1)$ matrix (a_{ij}) is invertible.

There is also a peculiar connection of projective varieties with affine varieties. A **projective variety** is defined similarly to an affine variety, except that the polynomials have to be homogeneous of some degree. This is necessary to have a well-defined zero-locus in projective space, which is obvious from homogeneous coordinates. The projective varieties can be thought of as affine varieties (”patches”) with additional, smaller spaces placed out at infinity. For instance, in $\mathbb{P}^1$, any point $(x : y)$ with $y \neq 0$ can be re-scaled to the form $(t : 1)$. Then $\mathbb{P}^1$ is just $\mathbb{C}$, and one more point $(1 : 0)$. We can think of $(1 : 0)$ as a point ”at infinity” that closes up the affine line $\mathbb{C}^1$ into the ”circle” $\mathbb{P}^1$.

Similarly, we can instead parameterize the points $(1 : t)$ by $t \in \mathbb{C}^1$ and have $(0 : 1)$ be the extra point. The subsets given by $\{(1 : t)\}$ and $\{(t : 1)\}$ are both called *affine patches* of $\mathbb{P}^1$, and form a cover of $\mathbb{P}^1$, from which we can inherit a natural topology on $\mathbb{P}^1$ from the Euclidean topology on each $\mathbb{C}^1$. In fact, the two affine patches form an open cover in this topology, so $\mathbb{P}^1$ is compact.

As another example, the projective plane $\mathbb{P}^2$ can be written as the disjoint union

$$\{(x:y:1)\} \cup \{(x:1:0)\} \cup \{(1:0:0)\} = \mathbb{C}^2 \cup \mathbb{C}^1 \cup \mathbb{C}^0,$$

which we can think of as a closure of the affine patch $\{(x:y:1)\}$, with a line and a point "at infinity". This will later be important in determining a cell structure on the projective space. The intersection of a projective variety with an affine patch gives us an affine variety, and the corresponding polynomial can be homogenized to carry this process out in reverse. The projective variety obtained in general in this way is called the **projective closure** of the corresponding affine variety. We can generalize this as follows.

Definition 7. *The* ***standard affine patches*** *of* $\mathbb{P}^n$ *are the sets*

$$\{(t_0 : t_1 : \cdots : t_{i-1} : 1 : t_{i+1} : \cdots : t_n) \mid t_i \in \mathbb{C}^n\}$$

for $i = 0, \ldots, n$.

Let us prove another surprising result, that all conics in $\mathbb{P}^2$ are the same, up to projective transformations.

Theorem 2.2.1. *All non-degenerate conics in* $\mathbb{P}^2$ *are the same, up to projective transformations.*

Proof. We have to show that, for any quadratic homogeneous polynomial $f(x, y, z)$ there is a projective transformation that sends it to one of x^2, $x^2 + y^2$, or $x^2 + y^2 + z^2$. Then , we can conclude that any two "non-degenerate" conics are the same up to a projective transformation. Any quadratic form can be written as $\mathbf{xAx}^T$ for a symmetric matrix $\mathbf{A}$, where $\mathbf{x}$ is a vector. By the spectral theorem, A is diagonalizable, that is $\mathbf{A}$ can be written as $\mathbf{A} = \mathbf{PDP}^T$ for some diagonal matrix $\mathbf{D}$. Then the quadratic form becomes $\mathbf{xAx}^T = \mathbf{xPDP}^T\mathbf{x}^T = (\mathbf{xP})\mathbf{D}(\mathbf{xP})^T$ which can be written as $\lambda_1(\mathbf{c}^T x)^2 + \lambda_2(\mathbf{c}^T x)^2 + \lambda_3(\mathbf{c}^T x)^2$ where λ_i is the entry in the i-th column of D and c_i is the i-th column vector of P. We can choose a projective transformation where the variables equal the terms inside the squares. □

The following definition will be important.

Definition 8. *An* $(n-1)$*-plane or* ***hyperplane*** *in* $\mathbb{P}^n$ *is the set of solutions* $(x_0 : \cdots : x_n)$ *to a homogeneous linear equation*

$$a_0 x_0 + a_1 x_1 + \cdots + a_n x_n = 0.$$

A k*-plane is an intersection of* $n-k$ *hyperplanes, say* $a_{i0}x_0 + a_{i1}x_1 + \cdots + a_{in}x_n = 0$ *for* $i = 1, \ldots, n-k$*, such that the matrix of coefficients* (a_{ij}) *is full rank.*

Intersection problems are generally much nicer in projective spaces. For example, for the first of our motivating questions, which asks how many points are contained in two distinct lines, the answer in $\mathbb{P}^2$ is always 1, since all lines intersect at a point, even parallel ones.

For the sake of completeness in the last chapter, I will formally define *degree* of a curve (or a variety in general). We will define the degree of a plane curve as the maximum possible intersection with a linear sub-variety (here, we consider plane curves as varieties in k^2).

Definition 9. *A linear sub-variety of $\mathbb{P}^n$ is a closed subset of a variety (in the Zariski Topology) defined by linear homogeneous polynomials.*

Then, an m-dimensional linear sub-variety of $\mathbb{P}^n$ is a projective sub-variety determined by a $(m+1)$-dimensional subspace of the vector space $\mathbb{C}^{n+1}$. Furthermore, the projective closure of a linear sub-variety of $\mathbb{A}^n$ is a linear sub-variety in $\mathbb{P}^n$.

Definition 10. *The degree of a variety $V \subset \mathbb{P}^n$, denoted by $deg(V)$, is defined as*

$$deg(V) := max\{\#(V \cap L) < \infty\}$$

where L is linear in $\mathbb{P}^n$ and $dimL + dimV = n$.

Thus, the degree of V is the number of points common to V and a $(codimV)$-dimensional "generic" linear sub-variety.

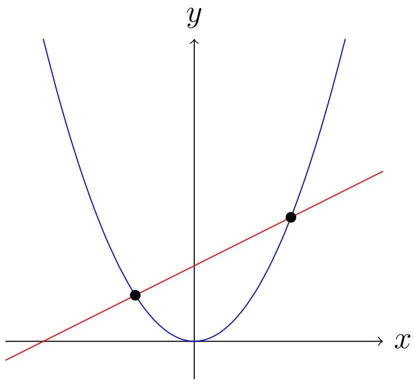


Figure 2.3: A generic line intersects a conic in two distinct points

We can then prove the following theorem, which ensures us that our two definitions of "degree" are consistent. [3]

Theorem 2.2.2. *If F is an irreducible homogeneous polynomial of degree d, then the degree of the hypersurface $\mathbb{V}(F) \subset \mathbb{P}^n$ is d.*

Proof. Give an arbitrary line $L \subset \mathbb{P}^n$, the intersection point of V and L can be identified with the zero of the polynomial function on L obtained by restricting F to the line L. The restriction of F to the line L produces a degree d polynomial on $L \cong \mathbb{C}$ which, by the Fundamental Theorem of Algebra, has d roots.

For a generic line L, these roots are distinct, and correspond to the d intersection points of V and L. □

2.3 Grassmannians

We can now define the Grassmannians, and show two directions of motivation. Firstly, a point in the projective space $\mathbb{P}^n$ is a line in the affine space of one higher dimension, i.e. $\mathbb{C}^{n+1}$. Then, a line in our projective space will be the **image** of a plane in $\mathbb{C}^{n+1}$. Rather than looking at intersection of k-planes in $\mathbb{P}^n$, we want to look at intersection of subspaces in the higher dimensional affine space.

Rather than just looking at 1-dimensional subspaces of $\mathbb{C}^n$, we want to generalize and consider say, k-dimensional subspaces.

Definition 11. *The **Grassmannian Gr(k,n)** is the set of all k-dimensional subspaces of $\mathbb{C}^n$.*

Note that $Gr(1, n+1)$ is exactly the projective space $\mathbb{P}^n$. Additionally, the set of all hyperplanes in a space is also identified with the same set. The Grassmannian also has the structure of a differentiable complex manifold, which we will not have the opportunity to discuss. We primarily care about the algebraic structure:

Theorem 2.3.1. *The Grassmannian $Gr(k, n)$ is a projective variety.*

Proof. Let $\Lambda \in \mathrm{Gr}(k, n)$ be a k-dimensional vector space in $\mathbb{C}^n$. Choose a basis vector $(a_{j1}, \ldots, a_{jn})$; $1 \leq j \leq k$ for Λ, and form the row matrix of basis vectors

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix}$$

This matrix has full rank, since its rows are linearly independent.
Two matrices of full rank (a_{ij}) and (b_{ij}) span the same subspace if and only if there exists $g \in \mathrm{GL}(k)$ satisfying $(a_{ij}) = g(b_{ij})$. Therefore, $\mathrm{Gr}(k, n)$ can be identified as

$$G = \{\text{matrices of size } k \times n \text{ and rank } k\} / \text{action of } \mathrm{GL}(k)$$

Let $\Delta_{(i_1,\ldots,i_k)} = k \times k$ subdeterminant of (a_{ij}) by columns for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.
The mapping

$$G \to \mathbb{P}^{\binom{n}{k}-1}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} \mapsto (\Delta_{(1,\ldots,k)} : \cdots : \Delta_{(i_1,\ldots,i_k)} : \cdots : \Delta_{(n-k+1,\ldots,n)})$$

is well-defined on G. Also, it is injective. It is known as the *Plücker embedding.* Under this identification, $\mathrm{Gr}(k, n)$ is a subset of $\mathbb{P}^{\binom{n}{k}-1}$. □

In the above proof, we choose an ordering on the k-element subsets S of $\{1, 2, \ldots, n\}$ and use this ordering to label the homogeneous coordinates x_S of $\mathbb{P}^{\binom{n}{k}-1}$. Now, given a point in the Grassmannian represented by a matrix M, let x_S be the determinant of the $k \times k$ submatrix determined by the columns in the subset S. This determines a point in projective space since row operations can only change the determinants up to a constant factor, and the coordinates cannot all be zero since the matrix has rank k.

Example 1. *In $Gr(4, 2)$, the matrix*

$$\begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & -3 & 0 & 3 \end{pmatrix}$$

has Plücker coordinates given by the determinants of all the 2×2 submatrices formed by choosing two of the columns above. We write x_{ij} for the determinant formed by columns i and j, so for instance,

$$x_{24} = \det \begin{pmatrix} 0 & 2 \\ -3 & 3 \end{pmatrix} = 6.$$

If we order the coordinates $(x_{12} : x_{13} : x_{14} : x_{23} : x_{24} : x_{34})$ then the image of the above point under the Plücker embedding is $(0 : -1 : -2 : 3 : 6 : 3)$.

The identification of the Grassmannian with full rank $k \times n$ matrices modulo row operations in the above proof will be crucial. This will let us define *Schubert Cells* in a very convenient way, and also give an easy way to find the *Plücker coordinates*, which we will see in an example. I will repeat the defintion of the Grassmannian once it becomes relevant in the last chapter. Another remark that should be made here is that once we consider subspaces, and the Grassmannian in particular, we are in effect translating our intersection problems to the origin, and since we are concerned with affine or projective geometry, this is not an issue. In fact, this is exactly the situation that we want, as our primary concern is the geometric structure, not the vector space structure specifically.

Chapter 3

The Combinatorics of Tableaux

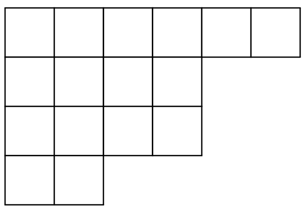
In this chapter we will focus on the purely combinatorial aspects of Young Tableaux. We will define a product on the tableaux which will lead to a Tableaux ring, and then explore connections to other combinatorial aspects, particularly symmetric polynomials. [1]

3.1 Young Diagrams

We first start with a series of definitions and examples to become familiar with Tableaux.

Definition 12. *A* ***Young Diagram*** *is a collection of boxes, or cells, arranged in left-justified rows, with a (weakly) decreasing number of boxes in each row.*

Example 2. *For example, the following is a young diagram:*



Each Young Diagram has an associated partition of the total number of boxes in the diagram, which is a tuple of the number of boxes in each row. This partition is also called the *shape* of the diagram, which we denote by λ. For example, the diagram in the example above has shape $\lambda = (6, 4, 4, 2)$. We can find the *size* of this partition by summing over all the number of boxes in each row, i.e. $|\lambda| = \sum_i \lambda_i$, which in this case is $6 + 4 + 4 + 2 = 16$.

Definition 13. *A* ***Young tableau****, or simply tableau, is a filling T of a young diagram that is*
(i) weakly increasing across each row;
(ii) strictly increasing down each column.

Definition 14. *A* ***standard tableau*** *is a tableau where the entries we use are numbers from 1 to n, where $n = |\lambda|$.*

Example 3. *Examples of Tableau:*

1	2	2	3	3	5
2	3	5	5		
4	4	6	6		
5	6				

Young Tableau

1	3	7	12	13	15
2	5	10	14		
4	8	11	16		
6	9				

Standard Tableau

We can use any **alphabet** (a totally ordered set) to form a filling of a Young diagram, but we usually use the positive integers.

Definition 15. *A* ***skew shape*** *is the difference ν/λ formed by cutting out the Young diagram of a partition λ from the strictly larger partition ν. A skew shape is a* ***horizontal strip*** *if no column contains more than one box.*

The skew shape will lead to the following important class of objects.

Definition 16. *A* ***semistandard Young tableau (SSYT)*** *of shape ν/λ is a Standard filling of the skew shape ν/λ. An SSYT has* ***content*** *μ if there are μ_i boxes labeled i for each i. The* ***reading word*** *of the tableau is the word formed by concatenating the rows from bottom to top.*

Example 4. *The following is a semistandard Young tableau of shape ν/λ and content μ where $\nu = (6, 5, 3)$, $\lambda = (3, 2)$, and $\mu = (4, 2, 2, 1)$. Its reading word is 134223111.*

			1	1	1
		2	2	3	
1	3	4			

We will use these skew shapes and SSYTs to eventually define the Schur polynomials, a very important class of symmetric polynomials. The Tableau defined above have a very deep combinatorial significance. For example, there is a well-known explicit formula, known as the Hook length formula, for the number of standard Young tableaux of a given shape. We will state the formula here without proof.

Definition 17. *For a square s in a Young diagram, define the hook length $hook(s) = arm(s) + leg(s) + 1$ where $arm(s)$ is the number of squares strictly to the right of s in its row and $leg(s)$ is the number of squares strictly below s in its column.*

Theorem 3.1.1 (**Hook length formula**)**.** *The number of standard Young tableaux of shape λ is*

$$\frac{|\lambda|!}{\prod_{s\in\lambda} hook(s)}$$

Example 5. *If $\lambda = (2, 2)$ then we have*

$$\frac{4!}{3 \cdot 2 \cdot 1} = 2$$

standard Young tableaux of shape λ.

3.2 The Schensted Bumping Algorithm

There are two fundamental operations that define a product on the Tableaux; the Schensted "bumping" algorithm, and the Schützenberger "sliding" algorithm. We will focus on the *bumping* algorithm, as most of the combinatorial properties of tableaux can be deduced from it.

- We first start with the simplest case of "row-insertion". We take a tableau T and a positive integer x, and construct a new tableau, denoted by $\mathbf{T} \leftarrow \mathbf{x}$. This tableau will have one more box than T, and its entries will be those of T together with one more entry labelled x.
- If x is at least as large as all the entries in the first row of T, simply add x in a new box to the end of the first row.
- If not, find the left-most entry in the first row that is strictly larger than x. Put x in the box of this entry, and remove ("bump") the entry.
- Take this entry that was bumped from the first row, and repeat the process on the second row.
- Keep going until the bumped entry can be put at the end of the row it is bumped into, or until it is bumped out the bottom, in which case it forms a new row with one entry.

This will become clear with the following example.

Example 6. *For example, we follow the algorithm if we want to insert* 2 *into the following Tableau* T*:*

1	2	2	3
2	3	5	5
4	4	6	
4	6		

Since 2 is not as large as all the entries in the first row because of the 3 at the end, the 2 bumps the 3 from the first row, which then bumps the first 5 from the second row, which bumps the 6 from the third row, which can be put at the end of the fourth row.

1	2	2	3
2	3	5	5
4	4	6	
5	6		

$\leftarrow 2$

1	2	2	2
2	3	5	5
4	4	6	
5	6		

$\leftarrow 3$

1	2	2	2
2	3	3	5
4	4	6	
5	6		

$\leftarrow 5$

1	2	2	2
2	3	3	5
4	4	5	
5	6	6	

$T \leftarrow 2$

It is clear from the construction that the result of this process is always a tableau. Each row is successively constructed to be weakly increasing, and, when an entry y bumps an entry z from a box in a given row, the entry below it, if there is one, is strictly larger than z (by the definition of a tableau), so z either stays in the same column or moves to the left, and the entry lying above its new position is no larger than y, so is strictly smaller than z. The process is also invertible. If we know the final tableau and the entry that was inserted, we can recover the original tableau, by simply running the algorithm backwards.

To generalize this process to find the product of two tableaux, say T and U, we simply find $T \cdot U$ by repeatedly applying the row-bumping algorithm to entries of U. Define w to be the reading word of U, say of length k. Recall that the reading word of a tableau is the entries of the tableau placed together in a sequence from left to right and bottom to top. We then run the algorithm $T \leftarrow w_i$ for $1 \leq i \leq k$, in that order. Then $T \cdot U$ has as many boxes as the sum of the boxes of T and U.

Example 7. *Consider the product*

$$\begin{array}{|c|c|c|c|}\hline 1&2&2&3\\\hline 2&3&5&5\\\hline 4&4&6\\\cline{1-3} 5&6\\\cline{1-2}\end{array} \cdot \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&2&2&2\\\hline 2&3&3&5\\\hline 4&4&5\\\cline{1-3} 5&6&6\\\cline{1-3}\end{array} \cdot \begin{array}{|c|c|}\hline 1&3\\\hline\end{array} = \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 2&2&3&5\\\hline 3&4&5\\\cline{1-3} 4&6&6\\\cline{1-3} 5\\\cline{1-1}\end{array} \cdot \begin{array}{|c|}\hline 3\\\hline\end{array} = \begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&3\\\hline 2&2&3&5\\\cline{1-4} 3&4&5\\\cline{1-3} 4&6&6\\\cline{1-3} 5\\\cline{1-1}\end{array}$$

We state without proof the following theorem:

Theorem 3.2.1. *The product operation makes the set of tableaux into an associative monoid. The empty tableau is a unit in this monoid:* $\phi \cdot T = T \cdot \phi = T$.

A **monoid** is a set with an associative binary operation and identity (e.g. the non-negative integers).

3.3 The Calculus of Tableaux

We now study the word of a tableau in more detail. Most of the crucial theorems about tableaux come from this examination.

Definition 18. *Words are all sequences of letters in an alphabet (the integers in our convention). The juxtaposition of two words is a binary operation which we write as* $w \cdot w'$.

Our primary goal will be to understand what the bumping algorithm does to the corresponding words of a tableau. We first notice that a tableau can be recovered from its word. We simply break the word wherever one entry is strictly larger than the next (in the language of combinatorics, these are called *descents*), and the blocks form the rows of the tableau, read in the same left-right, bottom-up convention as before. For example, the word 5644623551223 breaks into 56|446|2355|1223. It is also clear not all words have an associated tableau, since the blocks must have weakly increasing length (this is the young diagram requirement) and when they are arranged on top of each other, the columns must have strictly increasing entries. Different skew tableau can also determine the same word. Every word can in fact arise from a *skew* tableau (notice not a tableau) since we can break the word into increasing pieces and simply place the rows above and entirely right to the previous.

We now want to investigate what row-bumping will do, and how the word of a product of two tableaux is related to the words of its factors. Suppose an element x is row-inserted into a row. In "word" language, the Schensted algorithm says to factor the word of the row into $ux'v$, where u and v are words, x' is a letter, and each letter in u is no larger than x, and x' is strictly larger than x. The letter x' is to be replaced by x, so the row with word $ux'v$ becomes uxv, and x' is bumped to the next row. The resulting tableau has word $x'uxv$. Think of the example of 2 and 3 that we saw in the very first example of row-bumping. This is simply a formalism of that process. In the word code, the basic algorithm is

$$(ux'v) \cdot x \quad \mapsto \quad x'(uxv) \quad \text{if} \quad u \leq x \leq x' \leq v.$$

Here u and v are weakly increasing, and an inequality such as $u \leq v$ means that every letter in u is smaller than or equal to every letter in v. In this code, the row-insertion of 2 in the tableau with word 5644623551223 can be written

$$\begin{aligned}
&(5\,6) \cdot (4\,4\,6) \cdot (2\,3\,5\,5) \cdot (1\,2\,2\,3) \cdot 2 \mapsto (5\,6) \cdot (4\,4\,6) \cdot (2\,3\,5) \cdot (3 \cdot 5) \cdot (1\,2\,2\,2)\\
&\mapsto (5\,6) \cdot (4\,4\,5) \cdot (2\,3\,5) \cdot (1\,2\,2\,2)\\
&\mapsto (5\,6) \cdot (4\,4\,5) \cdot (2\,3\,3\,5) \cdot (1\,2\,2\,2)\\
&\mapsto (5\,6) \cdot (4\,4\,5) \cdot (2\,3\,3\,5) \cdot (1\,2\,2\,2).
\end{aligned}$$

Donald Knuth was able to study this process as a computer algorithm, which allowed a deeper insight into the structure and allowed the proofs of most of the theorems surrounding words. With a little thought into the formal process, we can in fact conclude a simple set of transformations, called **elementary Knuth transformations**, that govern this process. The following basic transformations on three consecutive letters are denoted by K' and K'' respectively:

$$\begin{array}{|c|c|}\hline y & z \\ \hline \end{array} \cdot \begin{array}{|c|}\hline x \\ \hline \end{array} = \begin{array}{|c|c|}\hline x & z \\ \hline y \\ \cline{1-1} \end{array} \qquad y\,z\,x \mapsto y\,x\,z \quad (x < y \leq z)$$

$$\begin{array}{|c|c|}\hline x & z \\ \hline \end{array} \cdot \begin{array}{|c|}\hline y \\ \hline \end{array} = \begin{array}{|c|c|}\hline x & y \\ \hline z \\ \cline{1-1} \end{array} \qquad x\,z\,y \mapsto z\,x\,y \quad (x \leq y < z)$$

The transformations K and K'' or their inverses are called the elementary Knuth transformations. We say that two words are *Knuth equivalent* if they can be obtained from one another by a sequence of these transformations, and we write $w \equiv w'$ to represent this equivalence. We have more or less shown the following:

Theorem 3.3.1. *For any tableau T and positive integer x,*

$$w(T \leftarrow x) \equiv w(T) \cdot x.$$

A corollary of this is the following: the word of a product of tableaux is the product of the words of the tableaux.

3.4 Words: The Plactic Monoid

Through a canonical construction that we will skip, and the theorem in the above section, we can show the following:

Theorem 3.4.1. *Every word is Knuth equivalent to the word of a unique tableau.*

We will now try and perform the obvious construction that these results beg, i.e. consider the Knuth equivalent classes of words as a set. We will show that the words modulo Knuth equivalence are the tableaux.

This formalization is due to Knuth, Lascoux, and Schützenberger. Let $M = M_m$ be the set of Knuth equivalence classes of words on our alphabet $[m] = \{1, \ldots, m\}$. The juxtaposition of words determines a product on this set, since if $w = w'$ and $v = v'$, then by definition $wv \equiv w'v' \equiv w' \cdot v \equiv w'v'$. This makes M into an associative monoid, with unit represented by the empty word $\varnothing$. More formally, the words form a free monoid F; the product is the juxtaposition we have been using; and the unit is the empty word $\varnothing$. The map from F to M that takes a word to its equivalence class is a homomorphism of monoids; $M = F/R$, where R is the equivalence relation generated by the Knuth relations (K') and (K''). M is called the **plactic monoid**. What we have done amounts to saying that the monoid of tableaux is isomorphic to the plactic monoid $M = F/R$. The associativity of the product is also obvious. We can now regard the monoid as the set of tableaux with the product defined above.

Every monoid has an associated "group ring" (we will skip the details here). For the monoid of tableaux with entries in $[m]$, we denote the corresponding ring by $\mathbb{R}[m]$, and call it the **tableau ring**. This is the free $\mathbb{Z}$-module with basis the tableaux with entries in the alphabet $[m]$, with multiplication determined by the multiplication of tableaux; it is an associative, but not commutative ring. There is a canonical homomorphism from $\mathbb{R}[m]$ onto the ring $\mathbb{Z}[x_1, \ldots, x_m]$ of polynomials that takes a tableau T to its monomial x^T, where x^T is the product of the variables x_i, each occurring as many times in x^T as i occurs in T. We will see a specific case of this construction in the next section.

3.5 Symmetric Polynomials

Consider the tableau ring, and define S_λ in the tableau ring $R_{[m]}$ to be the sum of all tableaux T of shape λ. The image of S_λ, in the polynomial ring is the Schur polynomial, denoted by $s_\lambda(x_1, ..., x_m)$.

Definition 19. *Given a semistandard Young tableau T of shape λ, define $x^T = x_1^{m_1} x_2^{m_2} \ldots$ where m_i is the number of i's in T. The **Schur polynomial** for a partition λ is defined by*

$$s_\lambda = \sum_T x^T$$

where the sum ranges over all SSYT's T of shape λ.

Let's see a concrete example of this.

Example 8. *For the partition $\lambda = (2, 1)$, the tableaux*

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \ldots$$

are a few of the infinitely many SSYT's of shape λ.
The corresponding Schur polynomial is then

$$s_\lambda = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + 2x_1 x_2 x_3 + \ldots$$

We now give the definition of symmetric functions. These are probably the most important functions in combinatorics.

Definition 20. *The (graded)* ***ring of symmetric functions*** $\Lambda_{\mathbb{C}}(x_1, x_2, ...)$ *is the ring of bounded-degree formal power series* $f \in \mathbb{C}[[x_1, x_2, ...]]$ *which are symmetric under permuting the variables, that is,* $f(x_1, x_2, ...) = f(x_{\pi(1)}, x_{\pi(1)}, ...)$ *for any permutation* $\pi : \mathbb{Z}^+ \to \mathbb{Z}^+$ *and* $deg(f) < \infty$*.*

Example 9. $x_1^2 + x_2^2 + x_3^2 + \ldots$ *is a symmetric function of degree 2.*

It can be shown that the Schur polynomials are symmetric, and in fact, they form a vector space basis for $\Lambda(x_1, x_2, \ldots)$ as λ ranges over all partitions. There exist general formulas that will tell us how to multiply two Schur polynomials, but we will currently focus on two specific cases, which are just consequences of properties of row-bumping. They are called **Pieri rules**, since they are the same as formulas found by Pieri for multiplying Schubert varieties in the intersection (cohomology) ring of a Grassmannian. This connection is precisely the subject-matter of this project, and the topic of the final chapter.

Let (p) and (1^p) be the Young diagrams with one row and one column of length p respectively. Then the Pieri formulas are:

$$S_\lambda \cdot S_{(p)} = \sum_{\mu} S_\mu,$$

the sum over all μ's that are obtained from λ by adding p boxes, with no two in the same column; and

$$S_\lambda \cdot S_{(1^p)} = \sum_{\mu} S_\mu,$$

the sum over all μ's that are obtained from λ by adding p boxes, with no two in the same row. These facts are translations of the fact that the product of a tableau T times a tableau V whose shape is a row (and respectively column) has the shape μ specified in the first formula (and respectively the second), and that any tableau U of this shape factors uniquely into such a product $U = T \cdot V$.

Now we can apply the homomorphism $T \mapsto x^T$ from $\mathbb{R}[m]$ to the polynomial ring and deduce from the Pieri formulas, the following formulas that tell us how to multiply a Schur polynomial with a complete symmetric polynomial (obtained from a row tableau) and with an elementary polynomial (obtained from a column tableau),

$$s_\lambda(x_1, \ldots, x_m) \cdot h_p(x_1, \ldots, x_m) = \sum_{\mu} s_\mu(x_1, \ldots, x_m),$$

where $h_p(x_1, \ldots, x_m)$ is the complete symmetric polynomial of degree p, and the sum is over all μ's that are obtained from λ by adding p boxes, with no two in the same column;

$$s_\lambda(x_1, \ldots, x_m) \cdot e_p(x_1, \ldots, x_m) = \sum_{\mu} s_\mu(x_1, \ldots, x_m),$$

where $e_p(x_1, \ldots, x_m)$ is the elementary polynomial of degree p, and the sum is over all μ's that are obtained from λ by adding p boxes, with no two in the same row.

3.6 The Littlewood-Richardson Rule

The rule to multiply two general Schur polynomials is called the Littlewood-Richardson rule. To state it, we need a few more definitions.

Definition 21. *A word is **Yamanouchi** if every suffix $w_k w_{k+1} \cdots w_n$ contains at least as many letters equal to i as $i+1$ for all i.*

Example 10. *For instance, the word 231211 is Yamanouchi, because the suffixes 1, 11, 211, 1211, 31211, and 231211 each contain at least as many 1's as 2's, and at least as many 2's as 3's.*

Definition 22. *A **Littlewood-Richardson tableau** is a semistandard Young tableau whose reading word is Yamanouchi.*

Example 11. *The following is a skew Littlewood-Richardson tableau:*

			1	1	1
		2	2	3	
1	3	4			

Definition 23. *A sequence of skew tableaux $T_1, T_2, \dots$ form a **chain** if their shapes do not overlap and*

$$T_1 \cup T_2 \cup \cdots \cup T_i$$

is a partition shape for all i.

We can now state the general Littlewood-Richardson rule. We will not include the proof, as the combinatorics is quite involved.

Theorem 3.6.1. *We have*

$$s_{\lambda(1)} \cdots s_{\lambda(m)} = \sum_{\nu} c^{\nu}_{\lambda(1),\dots,\lambda(m)} s_\nu,$$

where $c^{\nu}_{\lambda(1),\dots,\lambda(m)}$ is the number of chains of Littlewood-Richardson tableaux of contents $\lambda^{(i)}$ with total shape ν.

The following corollary follows from the above theorem ($m = 2$).

Corollary 3.6.1.1.

$$s_\lambda s_\mu = \sum_{\nu} c^{\nu}_{\lambda\mu} s_\nu,$$

where $c^{\nu}_{\lambda\mu}$ is the number of Littlewood-Richardson tableaux of skew shape ν/λ and content μ.

Proof. By Theorem 3.6.1, $c^{\nu}_{\lambda\mu}$ is the number of chains of two Littlewood-Richardson tableaux of content λ and μ with total shape ν. The first tableau of content λ is a straight shape tableau, so by the Yamanouchi reading word condition and the semistandard condition, the top row can only contain 1's. Continuing this reasoning inductively, it has only i's in its ith row for each i. Therefore the first tableau in the chain is the unique tableau of shape λ and content λ.
Thus the second tableau is a Littlewood-Richardson tableau of shape ν/λ and content μ, and the result follows. □

Chapter 4

Algebraic Topology and Cohomology Theory

4.1 Introduction

The primary development in this chapter will serve as a primer of the relevant topological notions - while the full depth of the subject is difficult to include with the necessary complexity, I will try to introduce the relevant notions through an overview of the subject of algebraic topology. A certain comfort with topology and algebra, especially the theory surrounding the fundamental group will be assumed.

The goal of the machinery in this chapter is concerned with the classification of spaces, under a topological notion of equivalence of *homeomorphism*. Two spaces X and Y are homeomorphic if a bijective function $f : X \to Y$ is continuous, and the inverse function $f^{-1} : Y \to X$ is also continuous.

Example 12. *The function $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) = \tan(\frac{\pi}{2}x)$ is a homeomorphism because it is a continuous bijection with a continuous inverse. This is clear from the following diagrams.*

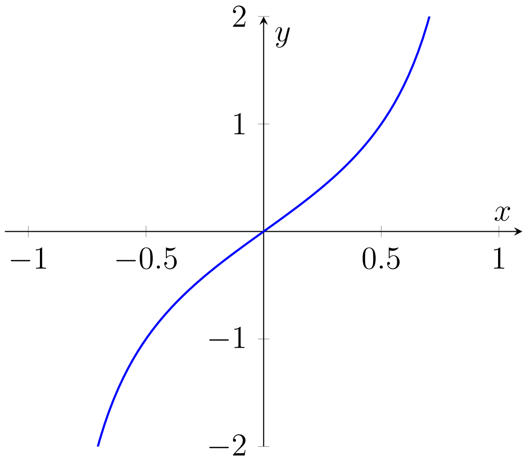


Figure 4.1: Tangent

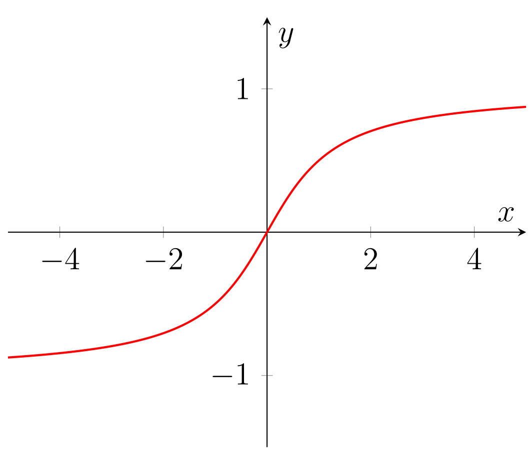


Figure 4.2: Inverse Tangent

In practice however, constructing such a continuous function to find a homeomorphism between two gives spaces is a very difficult problem, but many techniques have been developed for this purpose. To show that two spaces are *not* homeomorphic is an even more difficult task, for it requires us to show that such a continuous function *cannot* exist. Negative assertions like this usually require a more involved approach.

A usual practice is to find certain topological properties, which are "invariant" under continuous mappings - these allow us to differentiate between spaces immediately. The closed interval $[-1, 1]$ cannot be homeomorphic to the open interval $(-1, 1)$, because the closed interval is compact and the open is not, for instance. The line $\mathbb{R}$ cannot be homeomorphic to the plane $\mathbb{R}^2$ because deleting a point from $\mathbb{R}^2$ leaves a connected space but deleting a point from $\mathbb{R}$ does not. However, this is an approach with a limited scope. For more complicated spaces e.g. the 2-dimensional sphere S^2, the projective plane $\mathbb{P}^2$, the torus T, or the two-holed torus $T\#T$, it is difficult to find topological properties which may be preserved in this manner.

We wish to find more general objects which may be described as topological **invariants**. It turns out that there is category theoretic approach between Topological spaces and Algebraic objects (like groups, rings, modules etc.) that Algebraic Topology provides us which will be of primary interest to us. The objects which concern us will be *functors* that not only give us images of spaces, but also images of maps between these spaces as maps between the algebraic images.

4.2 The Fundamental Group

The first object of interest is the fundamental group [5] of a space, defined as follows:

Definition 24. *Let X be a space and x_0 be a point of X. $\pi_1(X, x_0)$ is the group of homotopy classes of loops based at x_0, under the group operation $*$ of path concatenation of classes (with the natural definition).*

By the above exposition, it will be in our interest to show that homeomorphic spaces have isomorphic fundamental groups. As a consequence of this theorem, we can show that if two spaces have distinct fundamental groups, then these spaces *cannot* be homeomorphic.

We would first like to remove the complication of fixing the base-point in X to find the fundamental group. It should be clear that this dependence is only on a path-component of X. Say there exists a path h from a point x_0 to another point x_1 of X. Then let $\bar{h}$ be the inverse path. It is obvious from construction that a loop based at x_1, say f, can be mapped to a loop at x_0 by the product $h * f * \bar{h}$. This is the idea of our proof.

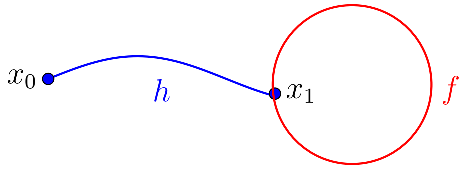


Figure 4.3: Change of base-point from x_1 to x_0

Theorem 4.2.1. *Define the map $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ by $\beta_h([f]) = [h * f * \bar{h}]$. Then, β_h is an isomorphism.*

Proof. We prove first that β_h is a homomorphism since $\beta_h[f * g] = [h * f * g * \bar{h}] = [h * f * \bar{h} * h * g * \bar{h}] = \beta_h[f]\beta_h[g]$. β_h is then an isomorphism with inverse $\beta_{\bar{h}}$ since $\beta_h\beta_{\bar{h}}[f] = \beta_h[\bar{h} * f * h] = [h * \bar{h} * f * h * \bar{h}] = [f]$, and similarly $\beta_{\bar{h}}\beta_h[f] = [f]$. □

Thus for a path connected space X, the fundamental group is unique up to isomorphism for every base-point, and we can simply write $\pi_1 X$ to refer to the fundamental group. We can assume this for a space for simplicity, without losing any generality for our purposes.

Definition 25. *A path-connected space X is simply connected if it has a trivial fundamental group.*

We can show easily that this definition is equivalent to saying that there is a unique homotopy class between any two points of X.

Example 13. *For a convex set X in $\mathbb{R}^n$ we have $\pi_1 X = 0$, the trivial group, since any two loops f_0 and f_1 based at x_0 are homotopic via the linear homotopy $f_t(s) = (1-t)f_0(s) + tf_1(s)$.*

To find non-zero fundamental groups is a more difficult task, and there is a considerable 'homotopy' theory surrounding the application of the fundamental group, concerned with the notions of covering spaces which allow us to compute fundamental groups, homotopy retracts, which allow us to find simpler spaces from a given space while not changing its fundamental group (the idea of homotopic equivalence of two spaces as compared to homeomorphism). The **Seifert-van Kampen Theorem** allows us to readily compute fundamental groups of complicated spaces.

The reader should consult standard texts like Munkres for a detailed exposition on these topics. I will share here the standard results that this theory allows:

Space	**Fundamental Group**
$\mathbb{R}^n$	0 (trivial group)
S^1	$\mathbb{Z}$
S^n (for $n \geq 2$)	0
$\mathbb{R}P^n$ (for $n \geq 2$)	$\mathbb{Z}/2\mathbb{Z}$
T (1-holed torus)	$\mathbb{Z}^2$
$\mathbb{C}P^n$	0
Figure-eight space	$\mathbb{Z} * \mathbb{Z}$ (free product of two copies of $\mathbb{Z}$)

Table 4.1: Spaces and their Fundamental Groups

We have been successful, more or less, in 'classifying' these spaces (more accurately, we have been able to differentiate between them). Any spaces that have distinct fundamental groups cannot be homeomorphic, which we will now show.

Definition 26. *Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. Define*

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

by the equation

$$h_*([f]) = [h \circ f].$$

The map h_* is called the homomorphism induced by h, *relative to the base point x_0.*

The map above is clearly well-defined.

The 'functorial' properties of the fundamental group are now clear, and the following results complete our task.

Theorem 4.2.2. *If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.*

Proof. The proof is a triviality. By definition,

$$(k \circ h)_*([f]) = [(k \circ h) \circ f],$$

$$(k_* \circ h_*)([f]) = k_*(h_*([f])) = k_*([h \circ f]) = [k \circ (h \circ f)].$$

Similarly, $i_*([f]) = [i \circ f] = [f]$. □

Corollary 4.2.2.1. *If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X with Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.*

Proof. Let $k : (Y, y_0) \to (X, x_0)$ be the inverse of h. Then $k_* \circ h_* = (k \circ h)_* = i_*$, where i is the identity map of (X, x_0); and $h_* \circ k_* = (h \circ k)_* = j_*$, where j is the identity map of (Y, y_0). Since i_* and j_* are the identity homomorphisms of the groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$, respectively, k_* is the inverse of h_*. □

4.3 CW Complexes

A very natural description of certain types of spaces will be very useful to us; these spaces are called **CW complexes** (also known as cell complexes or cellular complexes). The primary object of our discussion, the Grassmannian, has a cell complex description which will be extremely relevant. The motivation for CW complexes comes from a theorem known as **Adjoining a two-cell**, from which the definition follows easily. The theorem is reproduced here without proof for completeness. See Munkres for details.

Theorem 4.3.1 (Adjoining a two-cell)**.** *Let X be a Hausdorff space; let A be a closed path-connected subspace of X. Suppose that there is a continuous map $h : B^2 \to X$ that maps $\operatorname{Int} B^2$ bijectively onto $X - A$ and maps $S^1 = \operatorname{Bd} B^2$ into A. Let $p \in S^1$ and let $a = h(p)$; let $k : (S^1, p) \to (A, a)$ be the map obtained by restricting h. Then the homomorphism*

$$i_* : \pi_1(A, a) \to \pi_1(X, a)$$

induced by inclusion is surjective, and its kernel is the least normal subgroup of $\pi_1(A, a)$ containing the image of $k_* : \pi_1(S^1, p) \to \pi_1(A, a)$.

We will use this theorem to find the fundamental group of the torus in another, cleverer way. We first construct the torus as a certain well-known quotient space, then apply the theorem.

Theorem 4.3.2 (Munkres)**.** *The fundamental group of the torus has a presentation consisting of two generators α, β and a single relation $\alpha\beta\alpha^{-1}\beta^{-1}$.*

Proof. Let $X = S^1 \times S^1$ be the torus, and let $h : I^2 \to X$ be obtained by the standard quotient map that gives the torus as the identification space. Let p be the point $(0, 0)$ of $\operatorname{Bd} I^2$, let $a = h(p)$, and let $A = h(\operatorname{Bd} I^2)$. Then the hypotheses of the theorem above are satisfied.

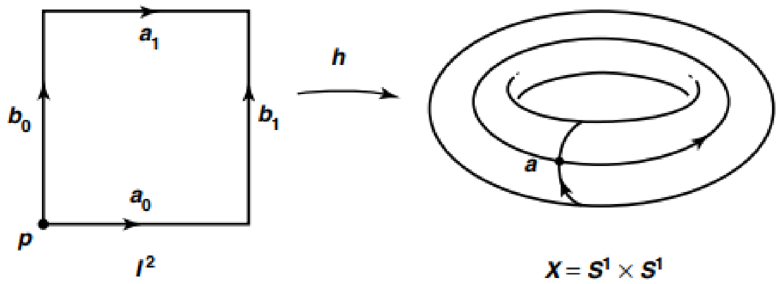


Figure 4.4: The Torus as a quotient space

The space A is the wedge of two circles, so the fundamental group of A is free. Indeed, if we let a_0 be the path $a_0(t) = (t, 0)$ and b_0 be the path $b_0(t) = (0, t)$ in $\mathrm{Bd}\, I^2$, then the paths $\alpha = h \circ a_0$ and $\beta = h \circ b_0$ are loops in A such that $[\alpha]$ and $[\beta]$ form a system of free generators for $\pi_1(A, a)$. See the figure below.

Now let a_1 and b_1 be the paths $a_1(t) = (t, 1)$ and $b_1(t) = (1, t)$ in $\mathrm{Bd}\, I^2$. Consider the loop f in $\mathrm{Bd}\, I^2$ defined by the equation

$$f = a_0 * (b_1 * (\overline{a_1} * \overline{b_0})).$$

Then f represents a generator of $\pi_1(\mathrm{Bd}\, I^2, p)$; and the loop $g = h \circ f$ equals the product $\alpha * (\beta * (\overline{\alpha} * \overline{\beta}))$. Adjoining a two-cell tells us that $\pi_1(X, a)$ is the quotient of the free group on the free generators $[\alpha]$ and $[\beta]$ by the least normal subgroup containing the element $[\alpha][\beta][\overline{\alpha}]^{-1}[\overline{\beta}]^{-1}$. □

Note that this simply means that the fundamental group of the torus is free on two generators, i.e. $F(a, b)$ modulo the commutator of the group (from the relation that $aba^{-1}b^{-1} = 1$). This in particular means that the fundamental group is abelian of rank 2.

The theory that is developed in this way can take us far - in fact, we can classify all compact surfaces up to homemorphism. This is done through a process of cutting and pasting, triangulation, and forming identification spaces, and these processes are closely related to the cellular description. I will give the larger result here. It can be shown that all compact surfaces can be constructed as in the statement of the theorem.

Theorem 4.3.3 (**The classification theorem**). *Let X be the quotient space obtained from a polygonal region in the plane by pasting its edges together in pairs. Then X is homeomorphic either to S^2, to the n-fold torus T_n, or to the m-fold projective plane P_m.*

The above description of the torus motivates us to consider all spaces that can be described inductively in this way. This turns out to be extremely useful as the Grassmannian will have this exact topological description. This structure will lead to certain abelian groups which form another class of topological invariants called homology groups.

We will consider now a constructive definition of a CW complex, and give a few examples. The fundamental building blocks of CW complexes are n-cells.

Definition 27. *An **n-cell** is a topological space homeomorphic to the open ball $|v| < 1$ in $\mathbb{R}^n$, and its associated n-disk is its closure $|v| \le 1$ in $\mathbb{R}^n$.*

We can construct a space by the following procedure: Start with a discrete set of points, and label this set X^0, the 0-skeleton. Attach 1-disks via continuous boundary maps from ∂D (which are points) to X^0. This is the 1-skeleton. These spaces are called *graphs*. Continue this process, inductively building the n-skeleton X^n, and set $X = X^n$.

To be more precise,

- We form X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$.
- Then X^n is the quotient space of the disjoint union $X^{n-1} \coprod_\alpha D^n_\alpha$ with the identification $x \sim \varphi(x)$ for $x \in \partial D^n_\alpha$.
- We then set $X^n = X^{n-1} \coprod_\alpha e^n_\alpha$, where each e^n_α is an open n-disk.
- If we stop at a finite stage $n < \infty$, then we set $X = X^n$ as our **CW complex**. If we continue indefinitely, then we set $X = \coprod_n X^n$, and X is given the weak topology: A set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n, and the topology on X^n is the usual Euclidean topology. The "C" stands for "closure-finite" and the W stands for "weak" topology.

Let's look at a few examples of CW complexes. The very last example will be directly relevant, as it is an example of the Grassmannian.

Example 14. *The sphere S^n has the structure of a cell complex with just two cells, e^0 and e^n, the n-cell being attached by the constant map $S^{n-1} \to e^0$. This is equivalent to regarding S^n as the quotient space $D^n/\partial D^n$.*

Example 15. *$\mathbb{RP}^n$ is the quotient space of a hemisphere D^n with antipodal points of ∂D^n identified. Since ∂D^n with antipodal points identified is just $\mathbb{RP}^{n-1}$, we see that $\mathbb{RP}^n$ is obtained from $\mathbb{RP}^{n-1}$ by attaching an n-cell, with the quotient projection $S^{n-1} \to \mathbb{RP}^n$ as the attaching map. It follows by induction on n that $\mathbb{RP}^n$ has a cell complex structure $e^0 \cup e^1 \cup \cdots \cup e^n$ with one cell e_i in each dimension $i \le n$. This is clear from the following diagram for example:*

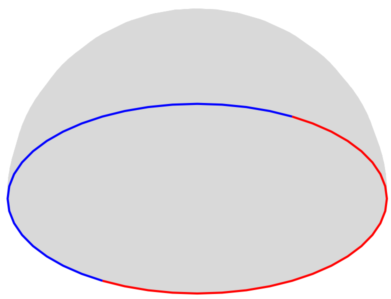


Figure 4.5: $\mathbb{RP}^2$ as the quotient of a sphere

Example 16. *The complex projective plane $\mathbb{CP}^2$ has a simpler cell complex structure, consisting of starting with a single point $X^0 = \{(1:0:0)\}$, and then attaching a 2-cell (a copy of $\mathbb{C} = \mathbb{R}^2$) like a balloon to form X^2. A copy of $\mathbb{C}^2 = \mathbb{R}^4$ is then attached to form X^4. These correspond to the line at infinity $\{(x:1:0)\}$ and the affine patch $\{(x:y:1)\}$ respectively.*

Example 17. *It is possible to generalize the affine patch closure construction for any n, where we will have (n+1) homogeneous coordinates. We will start with any affine patch, and then inductively attach the hyperplanes for each lower dimension. It follows that $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a cell e^{2n}. So by induction on n we obtain a cell structure $\mathbb{CP}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ with cells only in even dimensions.*

4.4 Homology Groups

Looking at the table of fundamental groups above, it is obvious that our task is naturally incomplete. Our "invariant" is unable to distinguish between certain higher dimensional spaces, e.g S^n for larger n, or the complex projective spaces. These are almost certainly not homeomorphic. A hint to why this happens is that the lower dimensional bias is in-built in the fundamental group: we were constructing homotopy classes from a 1-dimensional space into our space X, i.e. from the unit interval $I \to X$. We will see soon that in fact the fundamental group only depends on something called the 2-skeleton of X.

One natural way to overcome this limitation is to look at higher-dimensional homotopy groups denoted by $\pi_n(X)$ for a space, which are defined in terms of mapping the n-dimensional cube I^n into X, and the homotopies $I^n \times I \to X$ of such maps. When X is a CW complex, these groups depend on the $(n+1)$-skeleton of X. And indeed, they are able to distinguish between our previously 'indistinct' spaces. For example, for S^n, we get $\pi_i(S^n) = 0$ if $i < n$, and it is equal to $\mathbb{Z}$ for $i = n$.

There is however a significant challenge with the homotopy groups as their computations are exceedingly difficult (even in the simple example presented above. Thankfully, there are more computable groups that share some of these properties, and they are called the Homology groups, $H_n(X)$. The n-th Homology group also depends only on the $(n+1)$-skeleton of a space. In the example of S^n, the homology groups are isomorphic to the homotopy groups, but for $i > n$, we get the simplification that $H_i(S^n) = 0$. A famous theorem by Hurewicz connects these two classes of objects.

Theorem 4.4.1 (Hurewicz theorem). *For any path-connected space X and positive integer n, there exists a group homomorphism*

$$h_* : \pi_n(X) \to H_n(X),$$

called the Hurewicz homomorphism*, from the n-th homotopy group to the n-th homology group (with integer coefficients).*

The theorem in particular gives an isomorphism $\tilde{h}_* : \pi_1(X)/[\pi_1(X), \pi_1(X)] \to H_1(X)$ in the case of $n = 1$, which means the first homology group is the abelianization of the fundamental group.

The definition of homology groups is however much less transparent, and the full detail is beyond the scope of this paper. We will instead introduce homology through a particularly nice example, where the objects correspond to nice geometric descriptions. I will give the proper definitions, then hope for some elucidation through the example, taken directly from Hatcher. Then I will compute the groups for a few examples. This will lead naturally to cohomology, which will simply be a mirror of homology, and we will not get bogged down in the vast complexity of this theory. Note that our description uses the CW complex to compute homology, so it is rightfully called cellular homology. There is also a general theory of homology called singular homology, which for CW complexes is naturally equivalent. We will only concern ourselves with cellular homology.

Definition 28. *For a CW complex $X = X^n \supset \cdots \supset X^0$ define $C_k = Z^{\#\text{k-cells}}$ to be the free abelian group generated by the k-cells $B_\alpha^{(k)} = \left(D_\alpha^{(k)}\right)^o$.*

These are the chain groups and they will form the **chain complex**.

Definition 29. *The **cellular boundary map** $d_{k+1} : C_{k+1} \to C_k$ is $d_{k+1}(B_\alpha^{(k+1)}) = \sum_\beta deg_{\alpha\beta} \cdot B_\alpha^{(k)}$, where $deg_{\alpha\beta}$ is the degree of the composite map $\partial \overline{B_\alpha^{(k+1)}} \to X^k \to \overline{B_\beta^{(k)}}$.*

These boundary maps describe how cells attach to each other (in the inductive process). The first map above is the cellular attaching map from the boundary of the closure of the ball $B_\alpha^{(k+1)}$ to the k-skeleton, and the second map is the quotient map formed by collapsing $X^k \setminus B_\beta^{(k)}$ to a point. The composite is a map from a k-sphere to another k-sphere, which has a **degree**. We will not discuss this in detail.

It is known that the cellular boundary maps make the groups C_k into a **chain complex**, which is the following sequence of maps:

$$0 \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \longrightarrow 0$$

The most important property that boundary maps possess is that $\forall i \ d_i \circ d_{i+1} = 0$. This condition implies $im(d_{i+1}) \subset ker(d_i))$, which allows us to form quotient groups. These are the homology groups.

Definition 30. *The **cellular homology groups** of a space X are the abelian groups defined through the cellular chain complex above as such:*

$$H_i(X) = \frac{ker(d_i)}{im(d_{i+1})}$$

It is difficult to truly appreciate the nature of homology groups from the definitions alone, and we will now see an example (taken directly from Hatcher [4]) to make sense of what the attaching maps represent, and what the images and kernels of these maps represent, and why we take a quotient. The space we look at is the following:

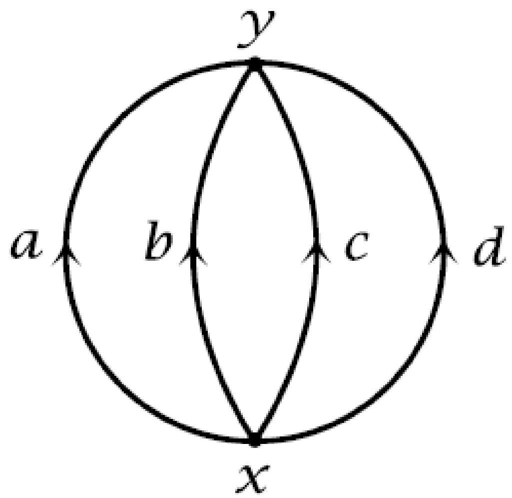


Figure 4.6: The space X

We have two vertices, labelled x and y, and four paths going from x to y, labelled a, b, c, and d.

The first natural consideration is to look at loops based at x. The fundamental group of X is clearly not abelian, and there are very complicated loops that can be formed, ranging from ab^{-1} to $ac^{-1}bd^{-1}ca^{-1}$. Suppose we simply matters by abelianizing. Then, for example, ab^{-1} and $b^{-1}a$ become identified with each other. These loops are actually the same "circle" just with different base-points, x and y respectively. This effectively causes our loops to become free of a base-point, as re-choosing a base-point is just a cyclic permutation of the "letters" in the word (in the sense of free groups). We now call our loops *cycles*.

Having abelianized, we can now switch to additive notation, and our cycles become linear combinations of the edges a, b, c, and d with integer coefficients, e.g. $a - b + c - d$. We can call these linear combinations *chains* of edges. Notice that some chains have multiple decompositions now that we have abelianized, e.g $(a - c) + (b - d) = (a - d) + (b - c)$. The immediate question we ask is that what are the possible "chains" that are actually meaningful, i.e. which chains have a geometric cycle interpretation as in our examples (e.g. $(a - b + a) = (2a - b)$ obviously cannot represent a cycle).

What condition do we have to impose on our chains for them to represent a cycle? A geometric cycle has the property that it enters a vertex exactly as many times as it leaves that vertex (in the non-example, the chain entered x once but left x twice). If we look at an arbitrary chain $ka + lb + mc + nd$, then the total number of times the chain enters y is $k + l + m + n$ (also the number of times it leaves x), since each of the edges enters y once. Thus, the total number of times it *enters* x is $-k - l - m - n$. These two numbers have to be equal which implies that $k + l + m + n = 0$ (the kernel condition!).

To generalize this to all graphs, let C_1 be the free abelian group with basis the edges i.e. $C_1 = F(a, b, c, d)$ and let C_0 be the free abelian group with basis the vertices i.e. $C_0 = F(x, y)$. Elements of C_1 are chains of edges, or 1-dimensional chains, and elements of C_0 are linear combinations of vertices, or 0-dimensional chains. Define a homomorphism $\partial : C_1 \to C_0$ by sending each basis element a, b, c, d to $y - x$, the vertex at the head of the edge minus the vertex at the tail (the images). Thus we have

$$\partial(ka + \ell b + mc + nd) = (k + \ell + m + n)y - (k + \ell + m + n)x,$$

and the cycles are precisely the kernel of ∂. It is a simple calculation to verify that $a - b$, $b - c$, and $c - d$ form a basis for this kernel. Thus every cycle in X_1 is a unique linear combination of these three most obvious cycles. By means of these three basic cycles we convey the geometric information that the graph X_1 has three visible 'holes', the empty spaces between the four edges.

Now what happens if we extend our cell complex to X^2 by attaching a 2-cell A to the cycle $a - b$, preserving orientation? Then this loop will no longer enclose a hole in X, and will be homotopically trivial, since we can now contract the path a onto b. Algebraically, we can quotient the cycles (kernel) with the subgroup generated by the cycle $a - b$. In the quotient, the cycles $a - d$ and $b - d$. are equivalent, since they are now homotopic. In this process, we also see the dependence on the cell structure.

Algebraically, we can define now a pair of homomorphisms $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ where C_2 is the infinite cyclic group generated by A and $\partial_2(A) = a - b$. The map ∂_1 is the boundary homomorphism in the previous example. The quotient group we are interested in is $\ker \partial_1 / \operatorname{Im} \partial_2$, the kernel of ∂_1 modulo the image of ∂_2, or in other words, the 1-dimensional cycles modulo those that are boundaries, the multiples of $a - b$. This quotient group is the homology group $H_1(X_2)$. The previous example can be fit into this scheme too by taking C_2 to be zero since there are no 2-cells in X_1, so in this case $H_1(X_1) = \ker \partial_1 / \operatorname{Im} \partial_2 = \ker \partial_1$, which as we saw was free abelian on three generators. In the present example, $H_1(X_2)$ is free abelian on two generators, $b - c$ and $c - d$, expressing the geometric fact that by filling in the 2-cell A we have reduced the number of 'holes' in our space from three to two.

We can generalize this process to attaching an n-cell, and define the n^{th} homology groups. This is what was formalized in the definitions. With this intuition, we can now compute the homology of some examples. Note that there is some additional structure on the Homology groups;

Example 18. *Since $\mathbb{CP}^2$ consists of a point, a 2-cell, and a 4-cell, its cellular chain complex becomes:*

$$\cdots \to 0 \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to 0$$

The homology groups are $H_0 = H_2 = H_4 = \mathbb{Z}$, $H_1 = H_3 = 0$. This has the obvious generalization.

Example 19. *For $\mathbb{RP}^2$, the chain complex looks like:*

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$$

The first map here is multiplication by 2 and the second is the zero map, from the degrees of the attaching maps. It follows that $H_2 = 0$, $H_1 = \mathbb{Z}/2\mathbb{Z}$ and $H_0 = 0$

Once we generalize to singular homology (in fact, through simplicial homology), we can prove a good number of results. We can look at some of these results without going into detail.

Example 20. *If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) = \mathbb{Z}$.*

Theorem 4.4.2. *The maps $f_* : H_n(X) \to H_n(Y)$ induced by a homotopy equivalence $f : X \to Y$ are isomorphisms for all n.*

This is the analog of a similar theorem we showed for the fundamental group. It shows the topological invariance of homology, and since they determine cohomology groups, of cohomology.

4.5 The Cohomology Ring

The cellular cohomology groups are an algebraic variant of homology, that result from dualizing our groups and our maps, which results in reversing the direction of our dual boundary maps. In terms of intrinsic information, nothing extra is present in the cohomology groups. In fact, homology groups determine the cohomology groups, and when the homology groups are finitely generated, the converse also holds. Let's state the definitions.

Definition 31. *The **dual** of the free abelian group C_k in the chain complex is $C^k = Hom(C_k, \mathbb{Z})$ for each k.*

$Hom(C_k, \mathbb{Z})$ is the group of homomorphisms from C_k to $\mathbb{Z}$

Definition 32. *The **coboundary map** $d_k^* : C^{k-1} \to C^k$ is defined by $d_k^* f(c) = f(d_k(c))$ for any $f \in C^k$ and any $c \in C_k$.*

The coboundary maps with the dual of the chain groups forms a **cochain complex**, and we can define the cohomology groups similarly.

Definition 33. *The i-th **cellular cohomology group** is the abelian group*

$$H^i(X) = \frac{ker(d_{i+1}^*)}{im(d_i^*}$$

Cohomology groups have a lot more structure than homology groups, due to contravariance. This is the primary distinction with homology groups, which are covariant functors. This leads to a natural product on the cohomology groups, called the **cup product**, which is the dual of the cap product on homology. The cup product is associative and distributive, and in fact, makes the cohomology groups into graded, commutative ring i.e. The direct sum of the cohomology groups

$$H^*(X) = \bigoplus H^i(X)$$

has a ring structure. The convention of calling a ring commutative actually corresponds to the product being anti-commutative i.e $\alpha \smile \beta = -(\beta \smile \alpha)$.

It is possible to view the cup product $\smile: H^p(X) \times H^q(X) \to H^{p+q}(X)$ as induced from the following composition:

$$C^\bullet(X) \times C^\bullet(X) \to C^\bullet(X \times X) \xrightarrow{\Delta^*} C^\bullet(X)$$

in terms of the chain complexes of X and $X \times X$, where the first map is known as the Künneth map and the second is the map induced by the diagonal $\Delta : X \to X \times X$.

A very famous result is the Poincare duality theorem.

Theorem 4.5.1 (Poincare Duality theorem). *If M is an n-dimensional oriented closed manifold (compact and without boundary), then for all integers k*

$$H^k(M) \cong H_{n-k}(M).$$

What relates this to our discussion is the fact that the cup product is Poincare dual to intersections. By taking the images of the fundamental homology classes of subspaces under inclusion, one can obtain a bilinear product on homology. We can state this as such,

Theorem 4.5.2. *Let M be an oriented smooth manifold of dimension n. If two submanifolds A, B of codimension i and j intersect transversely, then their intersection $A \cap B$ is again a submanifold of codimension $i + j$. Let $[A]^*, [B]^* \in H^i, H^j$ then there is the following equality:*

$$[A]^* \smile [B]^* = [A \cap B]^* \in H^{i+j}(X, \mathbb{Z}).$$

Here the notation $H^i(X, \mathbb{Z})$ refers to coefficient homology for general abelian groups, using $\mathbb{Z}$ which is the basic situation we have dealt with till now. The connections with Schubert Calculus (and what the conditions of being transverse translates to in our context for example) will become more apparent in the next chapter.

It is worth noting that there is an equivalent definition of cohomology on the Grassmannian known as the *Chow ring*, in which cohomology classes in $H^*(X)$ are equivalence classes of algebraic subvarieties under *birational equivalence*. In other words, deformations under rational families are still equivalent: in $\mathbb{P}^2$, for instance, the family of algebraic subvarieties of the form $xy - tz^2 = 0$ as $t \in \mathbb{C}$ varies are all in one equivalence class, even as $t \to 0$ and the hyperbola degenerates into two lines. This interpretation will be important in the next chapter.

Chapter 5

Intersection of Schubert Varieties in the Grassmannian

5.1 Schubert Cells

Let us first restate the definition of the Grassmannian, and use it to define Schubert cells as promised. [2]

Definition 34. *Each point of $Gr(k,n)$ is the row span of a unique full-rank $k \times n$ matrix in reduced row echelon form.*

This is because every point of the Grassmannian can be described as the span of some k independent row vectors of length n, which we can arrange in a $k \times n$ matrix. Then we can perform elementary row operations on the matrix without changing the row space, i.e. the point that it represents. See theorem 2.3.1 for details.

We will define the *Schubert cells* of a Grassmannian as all the points in the Grassmannian with a particular reduced row echelon form. We use the convention that the pivot entries are from left to right and top to bottom (our study of Young tableaux should hint at why we do this). Let us see this with an example.

Example 21. *The following matrix represents a point of $Gr(3,7)$:*

$$\begin{pmatrix} 0 & -1 & -3 & -1 & 6 & -4 & 5 \\ 0 & 1 & 3 & 2 & -7 & 6 & -5 \\ 0 & 0 & 0 & 2 & -2 & 4 & -2 \end{pmatrix}$$

It has the reduced row echelon form of the form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 1 & * & 0 & * & * & 0 \end{pmatrix},$$

where the $*$ is some complex number. These are all the entries that are not necessarily 1 *or* 0. *This is one of the Schubert cells of $Gr(3,7)$.*

Note that with this definition, it is also clear that the Grassmannian is the disjoint union of all the Schubert cells.

Eventually we want to construct something called *Schubert Varieties*. These have a structure of an algebraic variety, and an associated topology. Consider the second question we talked about in the introduction, asking how many lines intersect 4 given lines in 3-dimensional space:

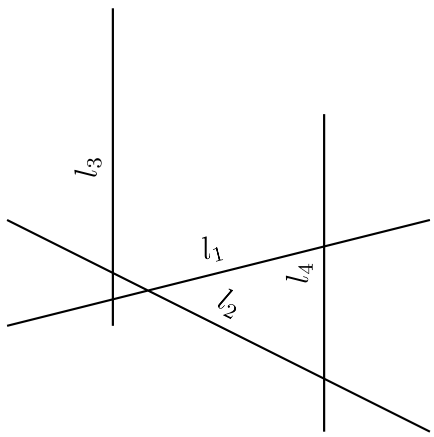


Figure 5.1: Four "general" lines in $\mathbb{R}^3$.

The original resolution by Schubert was to say that we can consider the case where two pairs of lines intersect one another. Then the line passing through these two intersection points is one solution, and each of the pairs of lines also determine a plane. Since we are in general position, these planes also intersect in a line, and this line also interests all of the four given lines. By continuity, this will always be the answer.

The intuition developed here can go astray very quickly, so we want to be more formal and rigorous. Our goal will be to construct the following sets:

$$\Omega_i = \{\text{All lines that pass through } l_i\}$$

And then compute the intersection $|\Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4|$ and show that the cardinality will always be 2.

First, we will try and enumerate the Schubert cells of the Grassmannian. It should be clear from the convention of row-reduced echelon forms that we used that we will accomplish this using tableaux. We first assign to the matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 1 & * & 0 & * & * & 0 \end{pmatrix}$$

a partition. Cut out the $k \times k$ staircase from the upper left corner of the matrix, and let λ_i be the distance from the edge of the staircase to the 1 in row i. In the example shown, we get the partition $\lambda = (4, 2, 1)$. Notice that we always have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$.

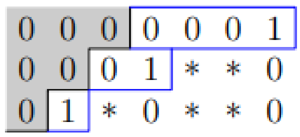
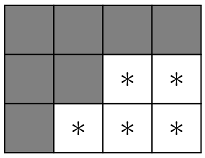


Figure 5.2: Cutting out the 3×3 staircase, and constructing λ

With this notation, Schubert cells in $Gr(k, n)$ are in bijection with the partitions λ for which $length(\lambda) \leq k$ and $\lambda_1 \leq n - k$. This is because we have k rows in total, and $n - k$ comes from the right-most pivot entry possible in row 1. Think about the largest values of λ_i's possible, and the corresponding arrangement of pivot entries. Then the Young diagram of our partition is inside the bounding shape of allowable partitions, which we call the **ambient rectangle**, of shape $k \times (n - k)$.

Example 22. *With our previous example of the Schubert cell, you get the shaded Young diagram in its ambient rectangle.*



Note that we can alternatively define λ as the complement in a $k \times (n - k)$ rectangle of the diagram μ defined by the right-aligned shift of the $*$ entries in the matrix.

Definition 35. *For a partition λ contained in the ambient rectangle, the* ***Schubert cell*** Ω_λ° *is the set of points of $Gr(n, k)$ whose row echelon matrix has corresponding partition λ. Explicitly,*

$$\Omega_\lambda^\circ = \{V \in Gr(n, k) \mid \dim(V \cap \langle e_1, \ldots, e_r \rangle) = i \text{ for } n-k+i-\lambda_i \leq r \leq n-k+i-\lambda_{i+1}\}.$$

Here e_{n-i+1} is the i-th standard unit vector $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 in the i-th position, so $e_1 = (0, 0, \ldots, 1)$, $e_2 = (0, 0, \ldots, 1, 0)$, and so on. The notation $\langle e_1, \ldots, e_r \rangle$ denotes the span of the vectors $e_1, \ldots, e_r$.

The numbers $n - k + i - \lambda_i$ are the positions of the pivot entries in the i-th row of the matrix, counted from the right. So, this condition says that the dimension of the intersection of V with the span of the standard basis vectors $\langle e_1, ..., e_r \rangle$ will be constantly equal to i from the time that e_r reaches the i -th pivot column, until it reaches the $(i + 1)$-th pivot column. We do not need to worry about index chasing in this definition exactly; we need only remember the above description of the Schubert cell.

Since each $*$ can be any complex number, we have $\Omega_\lambda^\circ = \mathbb{C}^{k(n-k)-|\lambda|}$ as a set, and so

$$\dim(\Omega_\lambda^\circ) = k(n - k) - |\lambda|.$$

In particular, the dimension of the Grassmannian is $k(n - k)$.

5.2 Schubert Varieties and Complete Flags

We can now define the *Schubert variety* as a closed subvariety of the Grassmannian. It is a kind of *moduli space* whose elements satisfy conditions giving lower bounds to dimensions of their intersection with specified subspaces.

Definition 36. *The* ***standard Schubert variety*** *corresponding to a partition λ, denoted Ω_λ, is the set*

$$\Omega_\lambda = \{V \in Gr(n, k) \mid \dim(V \cap \langle e_1, \ldots, e_{n-k+i-\lambda_i} \rangle) \geq i\}.$$

In the topology on the Grassmannian, as inherited from projective space via the Plücker embedding, the Schubert variety Ω_λ is the closure $\overline{\Omega_\lambda^\circ}$ of the corresponding Schubert cell. Note that we have $\dim(\Omega_\lambda) = \dim(\Omega_\lambda^\circ) = k(n-k) - |\lambda|$ as well.

We want to generalize our description to include other bases than the standard bases $e_1, \ldots, e_n$.

Definition 37. *A **complete flag** is a chain of subspaces*

$$F_\bullet : 0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n$$

A *partial flag* is a chain of subspaces in which only some of the possible dimensions are included. When $F_i = \langle e_1, \ldots, e_i \rangle$, then we say that $F_\bullet$ is the **standard flag**. We can also define the **opposite flag** $E_\bullet$ by setting $E_i = \langle e_n, \ldots, e_{n-i+1} \rangle$.

Definition 38. *Two subspaces V and W of $\mathbb{C}^n$ are **transverse** if*

$$\dim(V \cap W) = \max(0, \dim(V) + \dim(W) - n).$$

Equivalently, if $\operatorname{codim}(V)$ *is defined to be* $n - \dim(V)$*, then*

$$\operatorname{codim}(V \cap W) = \min(n, \operatorname{codim}(V) + \operatorname{codim}(W)).$$

We call a pair of flags F and E transverse if F_i and E_j are transverse for all i and j.

Example 23. *The standard and opposite flags are transverse trivially.*

There is actually a weaker condition under which two flags are transverse.

Theorem 5.2.1. *Two complete flags F and E in $\mathbb{C}^n$ are transverse if and only if $F_{n-i} \cap E_i = \{0\}$ for all i.*

Proof. (By Naughton [6]) In the forward direction, suppose that F and E are transverse. Then, for all i and j, we have

$$\dim(F_j \cap E_i) = \max(0, j + i - n).$$

Setting $j = n - i$ we can see that the dimension of this intersection is 0. Since $F_0 = E_0 = \{0\}$ the intersection $F_{n-i} \cap E_i$ is $\{0\}$.

In the backwards direction we induct on n. In the base case $n = 1$ we have that $F_i \cap E_j = \{0\}$ for i, j not both 1, while if i and j are both 1, we see $F_1 \cap E_1 = \mathbb{C} \cap \mathbb{C} = \mathbb{C}$. This satisfies the dimension conditions in Definition 38.

For induction we assume:

- If $F_{n-i-1} \cap E_i = \{0\}$ for all i, then F and E are transverse
- $F_{n-i} \cap E_i = \{0\}$ for all i.

In particular $F_{n-1} \cap E_1 = \{0\}$ and $\mathbb{C}^n = F_n = F_{n-1} \oplus E_1$. Now we can quotient both flags by E_1, reducing F_n to F_{n-1} and reducing the dimension of each E_i by 1 to get a new pair of flags

$$E' = \{0 = E_1/E_1 \subset E_2/E_1 \subset \cdots \subset E_n/E_1\}$$
$$F' = \{0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1}\}.$$

These new flags are transverse, since $F'_{n-i-1} \cap E'_i = F_{n-i-1} \cap E_{i+1}/E_1$. Since $F_{n-i-1} \cap E_{i+1} = \{0\}$ by assumption, we have $F'_{n-i-1} \cap E'_i = \{0\}$. By the inductive hypothesis F' and E' are transverse, so

$$\dim(F_i \cap E_j/E_1) = \max(0, i + j - 1 - (n - 1)) = \max(0, i + j - n).$$

Now we will show that F and E are transverse. For any $i \neq n$ we know $E_1 \not\subset F_i$, so

$$\dim(F_i \cap E_j) = \dim(F_i \cap E_j/E_1) = \max(0, i + j - n).$$

On the other hand, if $i = n$ and $j \neq 0$ then the intersection should have dimension j. Indeed, $E_1 \subset F_n$, so here we have

$$\dim(F_n \cap E_j) = 1 + \dim(F_{n-1} \cap E_j/E_1) = 1 + \max(0, n-1+j-n) = 1 + j - 1 = j.$$

The case where $i = n$ and $j = 0$ is trivial, as the intersection is simply $\{0\}$. Therefore, for all i and j we have

$$\dim(F_i \cap E_j) = \max(0, i + j - n).$$

This completes the proof. □

Another advantage of using flags is that we get to formalize Schubert's notion of "general" position. The *complete flag variety* $Fl(\mathbb{C}^n)$ is the collection of all complete flags in $\mathbb{C}^n$. It turns out that the matrices defining a complete flag are equivalent up to the action of the group B of upper triangular matrices. Further, $Fl(\mathbb{C}^n) \cong GL_n(\mathbb{C})/B$ has the structure of an algebraic variety. From this perspective we say that a property holds for a "general" collection of flags if it holds for all tuples of flags in some Zariski open dense subset of $Fl(\mathbb{C}) \times \cdots \times Fl(\mathbb{C})$.

We want to figure out intersections for arbitrary transverse flags and we can show that we only need to consider the standard and opposite flags for this purpose, because the general linear group GL_n acts transitively on complete flags. If $F_\bullet$ denotes the standard complete flag and V is any other complete flag, pick a basis for V so that $V_k = \langle v_1, \ldots, v_k \rangle$. Then the matrix $g \in GL_k$ whose columns are the vectors v_i will take F_k to V_k. Therefore, to understand the intersection of Schubert varieties with respect to transverse flags F' and E', we can do the computation using the standard and opposite flags and simply multiply the result by an appropriate $g \in GL_n$.

We can now define the general Schubert variety. The general Schubert cell is also defined similarly.

Definition 39. *The general* ***Schubert variety*** *corresponding to a partition λ, denoted Ω_λ, is the set*

$$\Omega_\lambda(F_\bullet) = \{V \in Gr(n, k) \mid \dim(V \cap F_{n-k+i-\lambda_i}) \geq i\}$$

Notice that if λ and μ are partitions with $\lambda_i \leq \mu_i$ for all i, then $\Omega_\lambda \supseteq \Omega_\mu$. In fact, Ω_λ is the disjoint union of all Schubert cells Ω°_μ where $\mu_i \geq \lambda_i$ for all i.

Example 24. *Consider the standard Schubert variety $\Omega_{\square\square}$ in $\mathbb{P}^5 = Gr(6, 1)$. The ambient rectangle is a 1×5 row of squares. There is one condition defining the points $V \in \Omega_{\square\square}$, namely $\dim(V \cap \langle e_1, e_2, e_3, e_4 \rangle) \geq 1$, where V is a one-dimensional* *subspace of* $\mathbb{C}^6$*. This means that* V *is contained in* $\langle e_1, \ldots, e_4 \rangle$*, and so, expressed in homogeneous coordinates, its first two entries (in positions* e_5 *and* e_6*) are 0. Thus each point of* $\Omega_{\square\square}$ *can be written in one of the following forms:*

$$\begin{array}{c}(0:0:1:*:*:*)\\(0:0:0:1:*:*)\\(0:0:0:0:1:*)\\(0:0:0:0:0:1)\end{array}$$

These are the Schubert cells and it follows that $\Omega_{\square\square}$ *can be written as a disjoint union of Schubert cells as follows:*

$$\Omega_{\square\square} = \Omega^{\circ}_{\square\square} \cup \Omega^{\circ}_{\square\square\square} \cup \Omega^{\circ}_{\square\square\square\square} \cup \Omega^{\circ}_{\square\square\square\square\square}$$

In fact, every Schubert variety is a disjoint union of Schubert cells.

We can also now translate our motivating example of how many lines intersect four given lines in $\mathbb{P}^3$ into the language of Schubert calculus as follows. We first consider a flag F in the higher-dimensional affine space $\mathbb{C}^4$. The variety $\Omega_1(F) \subset \mathrm{Gr}(4,2)$ consists of the two-dimensional subspaces V such that $\dim(V \cap F_{2+i-\lambda_i}) \geq i$ for all i. After some index chasing, this becomes the set of V with $\dim(V \cap F_2) \geq 1$ and $\dim(V \cap F_4) \geq 2$. Notice the second condition is trivially fulfilled in $\mathrm{Gr}(4,2)$, so $\Omega_1(F)$ is the set of planes in $\mathbb{C}^4$ intersecting the given plane F_2 in at least a line. We can now quotient $\mathbb{C}^4$ by scalar multiplication to obtain the set of lines in $\mathbb{P}^3$ intersecting a given line in at least a point.

Considering four flags F^1, F^2, F^3, F^4 in $\mathbb{C}^4$, the problem becomes equivalent to computing the intersection

$$\Omega_{\square}(F^1_\bullet) \cap \Omega_{\square}(F^2_\bullet) \cap \Omega_{\square}(F^3_\bullet) \cap \Omega_{\square}(F^4_\bullet).$$

It will also be helpful to work out the Schubert varieties in the extreme cases, first, when the partition is empty i.e. $\lambda = (0, 0, \ldots, 0)$ and second, when the partition occupies the entire ambient rectangle B i.e. $\lambda = (n-k, n-k, \ldots, n-k)$. In the former case, the dimension condition that $\dim(V \cap \langle e_1, \ldots, e_{n-k+i} \rangle) \geq i$ is trivially fulfilled since V is k-dimensional and $k+(n-k+i) = n+i$, and in $\mathbb{C}^n$ this guarantees the intersection has dimension at least i, so that $\Omega_\phi = Gr(k,n)$. In the latter case we see Ω_B is a single point in the Grassmannian, namely, the span of the first k basis vectors.

Now that we have obtained the Schubert varieties that we required, we want to understand how to carry out these products. The rule to accomplish this is called the **Littlewood-Richardson rule** (now through a geometric and equivalent lens). The Littlewood-Richardson rule is particularly nice in the case of zero-dimensional intersections. In particular, given a list of generic flags $F^{(i)}_\bullet$ in $\mathbb{C}^n$ for $i = 1, \ldots, r$, let $\lambda^{(1)}, \ldots, \lambda^{(r)}$ be partitions with

$$\sum |\lambda^{(i)}| = k(n-k).$$

Then the intersection

$$\bigcap \Omega_{\lambda^{(i)}}(F^{(i)}_\bullet)$$

is zero-dimensional, consisting of exactly $c^B_{\lambda^{(1)},\ldots,\lambda^{(r)}}$ points of $\mathrm{Gr}(n,k)$, where B is the ambient rectangle and $c^B_{\lambda^{(1)},\ldots,\lambda^{(r)}}$ is a certain **Littlewood-Richardson coefficient**.

5.3 Cup Product is Poincare Dual to Intersections

As the name suggests, the Schubert cells give the Grassmannian its cellular (CW complex) structure. We will not prove this fact here, but give the relevant construction.

Define the 0-skeleton X^0 to be the 0-dimensional Schubert variety $\Omega_{((n-k)^k)}$. Define X^2 to be X^0 along with the 2-cell (since we are working over $\mathbb{C}$ and not $\mathbb{R}$) given by $\Omega^\circ_{((n-k)^{k-1}, n-k-1)}$, and the attaching map given by the closure in $Gr(k,n)$. Note that the partition in this step is formed by removing a single corner square from the ambient rectangle.

Then, X^4 is formed by attaching the two four-cells given by removing two outer corner squares in both possible ways, giving either $\Omega^\circ_{((n-k)^{k-2}, n-k-1, n-k-1)}$ or $\Omega^\circ_{((n-k)^{k-1}, n-k-2)}$. We can continue in this manner with each partition size to define the entire cell structure, $X^0 \subset X^2 \subset \cdots \subset X^{2k(n-k)}$.

Example 25. *We have*

$$Gr(4,2) = \Omega^\circ_{(2,2)} \sqcup \Omega^\circ_{(2,1)} \sqcup \Omega^\circ_{(2)} \sqcup \Omega^\circ_{(1,1)} \sqcup \Omega^\circ_{(1)} \sqcup \Omega^\circ_{\phi},$$

forming a cell complex structure in which $X^0 = \Omega^\circ_{(2,2)}$, X^2 *is formed by attaching* $\Omega^\circ_{(2,1)}$, X^4 *is formed by attaching* $\Omega^\circ_{(2)} \sqcup \Omega^\circ_{(1,1)}$, X^6 *is formed by attaching* $\Omega^\circ_{(1)}$, *and* X^8 *is formed by attaching* Ω°_{ϕ}.

The CW structure and the connections with the cohomology (intersection) ring will help us prove the general zero-dimensional Littlewood-Richardson rule and compute the Littlewood-Richardson coefficients. The main fact that we presented in a different form in the last chapter can now be stated (without proof) more explicitly.

Theorem 5.3.1. *The cohomology ring* $H^*(\mathrm{Gr}(k,n))$ *has a* $\mathbb{Z}$*-basis given by the classes*

$$\sigma_\lambda := [\Omega_\lambda(F_\bullet)] \in H^{2|\lambda|}(\mathrm{Gr}(k,n))$$

for λ *a partition fitting inside the ambient rectangle. The cohomology* $H^*(\mathrm{Gr}(k,n))$ *is a graded ring, so* $\sigma_\lambda \cdot \sigma_\mu \in H^{2|\lambda|+2|\mu|}(\mathrm{Gr}(k,n))$, *and we have*

$$\sigma^\lambda \cdot \sigma^\mu = [\Omega_\lambda(F_\bullet) \cap \Omega_\mu(E_\bullet)]$$

where $F_\bullet$ *and* $E_\bullet$ *are the standard and opposite flags.*

The equivalence relation is exactly the birational equivalence that we had stated earlier. Also note that σ_λ is independent of the choice of flag $F_\bullet$, since any two Schubert varieties of the same partition shape are rationally equivalent via a change of basis.

We can now restate the intersection problems in terms of multiplying Schubert classes. In particular, if $\lambda^{(1)}, \ldots, \lambda^{(r)}$ are partitions with $\sum_i |\lambda^{(i)}| = k(n-k)$, then

$$\sigma_{\lambda^{(1)}} \cdots \sigma_{\lambda^{(r)}} \in H^{k(n-k)}(Gr(k,n)),$$

and there is only one generator of the top cohomology group, namely σ_B where B is the ambient rectangle. This is the cohomology class of the single point $\Omega_B(F_\bullet)$

for some flag $F_\bullet$. Thus the intersection of the Schubert varieties $\Omega_{\lambda^{(1)}}(F_\bullet^{(1)}) \cap \cdots \cap \Omega_{\lambda^{(r)}}(F_\bullet^{(r)})$ is rationally equivalent to a finite union of points, the number of which is the coefficient $c^B_{\lambda^{(1)},\ldots,\lambda^{(r)}}$ in the expansion

$$\sigma_{\lambda^{(1)}} \cdots \sigma_{\lambda^{(r)}} = c^B_{\lambda^{(1)},\ldots,\lambda^{(r)}} \sigma_B.$$

For a sufficiently general choice of flags, the $c^B_{\lambda^{(1)},\ldots,\lambda^{(r)}}$ points in the intersection are distinct with no multiplicity. In general, we wish to understand the coefficients that we get upon multiplying Schubert classes and expressing the product back in the basis $\{\sigma_\lambda\}$ of Schubert classes.

We say in the last section that the motivating question was reduced to computing the cardinality of the following intersection

$$\Omega_\square(F_\bullet^1) \cap \Omega_\square(F_\bullet^2) \cap \Omega_\square(F_\bullet^3) \cap \Omega_\square(F_\bullet^4).$$

for a generic choice of flags. Now we can further reduce the problem to computing the coefficient c for which

$$\sigma_\square \cdot \sigma_\square \cdot \sigma_\square \cdot \sigma_\square = c \cdot \sigma_{\boxplus}$$

in $H^*(Gr(2,4))$.

5.4 The Isomorphism

The following theorem will allow us to multiply cohomology classes by multiplying Schur polynomials.

Theorem 5.4.1. *There is a ring isomorphism*

$$H^*(G(k,n)) \cong \Lambda(x_1, x_2, \ldots)/(s_\lambda | \lambda \not\subset B)$$

where B is the ambient rectangle and $(s_\lambda|\lambda \not\subset B)$ is the ideal generated by the Schur functions. The isomorphism sends the Schubert class σ_λ to the Schur function s_λ.

Proof Sketch. First note that sending σ_λ to s_λ is an isomorphism of the underlying vector spaces, since on the right hand side we quotient by the Schur functions whose partition does not fit inside the ambient rectangle. So, it remains to show that this isomorphism respects the multiplications in these rings, taking cup product to polynomial multiplication. It is sufficient to show that the *Pieri rule* for Schur polynomials also holds for the cohomology classes because the Schur functions $s_{(r)}$ and corresponding Schubert classes $\sigma_{(r)}$ form an algebraic set of generators for their respective rings. Therefore, to intersect Schubert classes we simply have to understand how to multiply Schur functions. (the Pieri formula is exactly identical to the one we have already provided in Chapter 3, but with the added condition that the sum is only over the partitions that fit inside the ambient rectangle). □

We can show using the Duality Theorem that the Pieri rule for Schubert varieties is equivalent to the following statement, by multiplying both sides of the Pieri formula for cohomology classes (which is simply the Schur polynomial formula) by σ_{μ^c} to extract the coefficient of σ_μ on the right hand side:

Theorem 5.4.2 (Pieri Rule). *Let λ and μ be partitions with $|\lambda|+|\mu| = k(n-k)-r$. Then if $F_\bullet, E_\bullet$, and $H_\bullet$ are three sufficiently general flags then the intersection*

$$\Omega_\lambda(F_\bullet) \cap \Omega_\mu(E_\bullet) \cap \Omega_{(r)}(H_\bullet)$$

has 1 element if μ^c/λ is a horizontal strip, and it is empty otherwise.

We now finally return to the Littlewood-Richardson rule. As a consequence of Theorem 3.6.1 and Theorem 5.3.1, in $H^*(Gr(k,n))$ we have

$$\sigma_{\lambda^{(1)}} \cdots \sigma_{\lambda^{(m)}} = \sum_\nu c^\nu_{\lambda^{(1)},\ldots,\lambda^{(m)}} \sigma_\nu$$

where now the sum on the right is restricted to partitions ν fitting inside the ambient rectangle. Note that by the combinatorics of the Littlewood-Richardson rule, the coefficients on the right are nonzero only if $|\nu| = \sum |\lambda^{(i)}|$, and so in the case of a zero-dimensional intersection of Schubert varieties, the only possible ν on the right-hand side is the ambient rectangle B itself. Moreover, $\Omega_B(F_\bullet)$ is a single point of $Gr(k,n)$ for any flag $F_\bullet$. The zero-dimensional Littlewood-Richardson rule follows as a corollary.

Theorem 5.4.3 (Zero-Dimensional Littlewood-Richardson Rule). *Let B be the $k \times (n-k)$ ambient rectangle, and let $\lambda^{(1)}, \ldots, \lambda^{(m)}$ be partitions fitting inside B such that $|B| = \sum |\lambda^{(i)}|$. Also let $F^{(1)}_\bullet, \ldots, F^{(m)}_\bullet$ be any m generic flags. Then*

$$c^B_{\lambda^{(1)},\ldots,\lambda^{(m)}} := \left|\Omega_{\lambda^{(1)}}(F^{(1)}_\bullet) \cap \cdots \cap \Omega_{\lambda^{(m)}}(F^{(m)}_\bullet)\right|$$

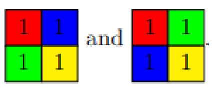
is equal to the number of chains of Littlewood-Richardson tableaux of contents $\lambda^{(1)}, \ldots, \lambda^{(m)}$ with total shape equal to B.

We have finally reached the point of a full solution to our motivating problem. I will restate the question here: Given 4 lines in general position in 3-dimensional space, how many lines intersect all 4 non-trivially?

We showed earlier that it suffices to compute the coefficient c in the expansion

$$\sigma_\square \cdot \sigma_\square \cdot \sigma_\square \cdot \sigma_\square = c \cdot \sigma_{\boxplus}$$

in $H^*(Gr(2,4))$. This is the Littlewood-Richardson coefficient $c^{(2,2)}_{\square,\square,\square,\square}$. This is the number of ways to fill a 2×2 ambient rectangle with a chain of Littlewood-Richardson tableaux having one box each.



There are **2 ways** to do this. Because each tableau in the chain must contain a single 1, we have colored the boxes to differentiate between different tableau in the chains. The condition that $T_1 \cup \cdots \cup T_i$ be a partition shape for all i necessitates that T_1, the red tableau, is always in the top left corner. The blue tableau, T_2 can then be placed in either the top right or bottom left corner, with the green tableau T_3 taking the other corner. Finally, T_4, the yellow tableau, must be placed in the bottom right corner. Therefore, the two fillings of a 2×2 ambient rectangle by chains of Littlewood-Richardson tableaux shown above are unique. Thus we have obtained the same answer as Schubert!

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